Total Restrained Detour Domination Number of a Graph

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Abstract: Let $G = (V, E)$ be a simple graph. A detour dominating set S of $V(G)$ is called a total restrained detour dominating set of G and if the subgraph induced by S and $V - S$ has no isolated vertices. The minimum cardinality of a total restrained detour dominating set of G is called the total restrained detour domination number of G and is denoted $by\, \gamma_{trd}(G)$. This paper, introduces the concept of total restrained detour domination number of graphs. Also, this number is found for some standard graphs.

Key words: detour domination number, total restrained detour domination number.

AMS Subject Classification: 05C78.

1. Introduction: The graphs considered here are finite and without loops and multiple edges. Detour dominating graphs were introduced and studied by J. John and N. Arianayagam[10]. For underlying definition & results, see references. The following results are from[10]

Theorem 1.1. For a non-trivial tree, $dn(G) = k$, where k is the number of end vertices of G.

Theorem1.2. $\gamma_d(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if\ n \geq 5 \\ 2 & if\ n = 2, 3\ or\ 4 \end{cases}$

Theorem1.3. For $if\ n \geq 6, \gamma_d(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem 1.4. For the complete graph $K_n (n \geq 2)$, $\gamma_d(K_n) = 2$.

Theorem 1.5. For the complete bipartite graph $G = K_{m,n}$,

$$\gamma_d(G) = \begin{cases} 2 & if\ m = n\ and\ m = 2, n \geq 3 \\ n - 1 & if\ \ m = 1, n \geq 2 \\ 3 & if\ \ 3 \leq m \leq n \end{cases}$$

Theorem 1.6. Every end vertex of G belongs to every detour dominating graph.

2. Total Restrained Detour Domination Number of a Graph

Definition 2.1 Let $G = (V, E)$ be a simple graph. A detour dominating set S of $V(G)$ is called a total restrained detour dominating set of G and if the subgraph induced by S and $V - S$ has no isolated vertices. The minimum cardinality of a total restrained detour dominating set of G is called the total restrained detour domination number of G and is denoted by $\gamma_{trd}(G)$.

Example 2.2.

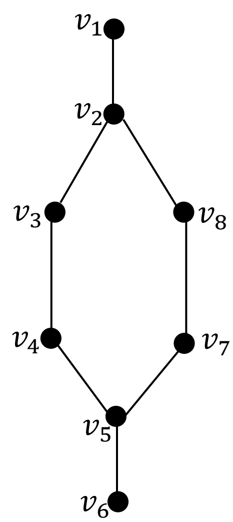

Figure 2.1

Here, $S = \{v_1, v_2, v_5, v_6\}$ is a unique minimum total restrained detour dominating set of G. Also, $\gamma_d(G) = \gamma_{trd}(G) = 4$.

Remark 2.3. A detour dominating set need not be a total restrained detour dominating set.

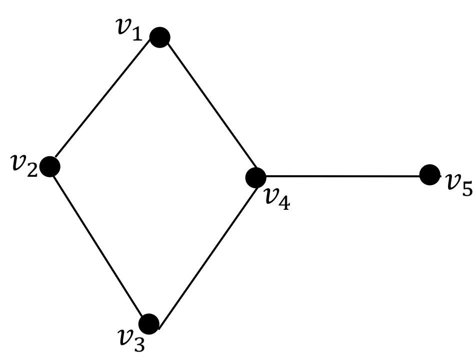

Figure 2.2

For example, $S = \{v_2, v_5\}$ is a detour dominating set of G in figure 2.2. But, S is not a total restrained detour dominating set of G.

Observation 2.4.

(i) For any connected graph G of order p, $2 \leq \gamma_d(G) \leq \gamma_{rd}(G) \leq \gamma_{trd}(G) \leq p$.

(ii) Every end vertex belongs to every total restrained detour dominating set of G.

Theorem 2.5. Let G be a connected graph and uv be a pendent edge of G. Then $\{u, v\}$ is a subset of every total restrained detour dominating set of G.

Proof. Let G be a connected graph. Let uv be a pendent edge of G with v as the end vertex.
Let S be a total restrained detour dominating set of G. By 2.4, $v \in S$.
Suppose $u \notin S$, then v is an isolated vertex in the subgraph induced by S,
which is a contradiction to S is a total restrained detour dominating set of G.
Therefore, $u \in S$.
Hence, $\{u, v\}$ is a subset of every total restrained detour dominating set of G.

Theorem 2.6. $\gamma_{trd}(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2 & if\ n = 4k\ or\ n = 4k+1 \\ \left\lceil \frac{n}{2} \right\rceil + 1 & if\ n = 4k+2\ or\ n = 4k+3. \end{cases}$

Proof. Case(i) Let $n = 4k$.

Let $V(P_n) = \{v_1, v_2, v_3, \dots, v_{4k-2}, v_{4k-1}, v_{4k}\}$. Choose the first two of every four vertices of $V(P_n)$ starting from v_1 to get a minimum total restrained detour dominating set of P_n .

Since, v_{4k} is an end vertex of P_n, every total restrained detour dominating set contains $\{v_{4k-1}, v_{4k}\}$.

Hence, $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\} \cup \{v_{4k-1}, v_{4k}\}$ is a minimum total restrained detour dominating set of P_n.

Therefore, $\gamma_{trd}(P_n) = |S| = 2k + 2$

$$= \left\lceil \frac{n}{2} \right\rceil + 2 \text{ where } n = 4k.$$

Case (ii): $n = 4k + 1$.

Let $V(P_n) = \{v_1, v_2, v_3, \dots, v_{4k-2}, v_{4k-1}, v_{4k}, v_{4k+1}\}$.

Proceeding as before to get a minimum total restrained detour dominating set of P_n, choose the first two of every four vertices of $V(P_n)$ starting from v_1.

Since, v_{4k+1} is an end vertex of P_n, every total restrained detour dominating set contains $\{v_{4k}, v_{4k+1}\}$.

Let $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\} \cup \{v_{4k}, v_{4k+1}\}$.

Then $G[V - S]$ has an isolated vertex v_{4k-1}.

Now, $S' = S \cup \{v_{4k-1}\}$ is a minimum total restrained detour dominating set of P_n.
Hence, $\gamma_{trd}(G) = |S'| = 2k + 3$
$= (2k + 1) + 2$
$= \left\lceil \frac{n}{2} \right\rceil + 2$ where $n = 4k + 1$.

Case (iii): $n = 4k + 2$.

Let $V(P_n) = \{v_1, v_2, v_3, \dots, v_{4k-1}, v_{4k}, v_{4k+1}, v_{4k+2}\}$.

Proceeding as before $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\} \cup \{v_{4k+1}, v_{4k+2}\}$ is a minimum total restrained detour dominating set of P_n.

Hence, $\gamma_{trd}(P_n) = |S|$
$= 2k + 2$
$= (2k + 1) + 1$
$= \left\lceil \frac{n}{2} \right\rceil + 1$ where $n = 4k + 2$.

Case (iv): $n = 4k + 3$.

Let $V(P_n) = \{v_1, v_2, v_3, \dots, v_{4k}, v_{4k+1}, v_{4k+2}, v_{4k+3}\}$. Proceeding as before starting from v_1, till we reach v_{4k}.

Hence, $\{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\}$ is a subset of every total restrained detour dominating set of P_n.

Now, the left out vertices are $\{v_{4k+1}, v_{4k+2}, v_{4k+3}\}$.

Here, v_{4k+3} is an end vertex. Therefore, $\{v_{4k+2}, v_{4k+3}\}$ is a subset of every total detour dominating set of P_n.

Now, $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\} \cup \{v_{4k+2}, v_{4k+3}\}$ is not dominating the vertex v_{4k}.

Further, $S \cup \{v_{4k+1}\}$ is a total restrained detour dominating sets of P_n and is also minimum.

Hence, $\gamma_{trd}(P_n) = |S| + 1 = 2k + 3$
$= (2k + 2) + 1$
$= \left\lceil \frac{n}{2} \right\rceil + 1$ where $n = 4k + 3$.

Theorem 2.7. $\gamma_{trd}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & if \quad n = 4k \ or \ 4k + 1 \\ \left\lceil \frac{n}{2} \right\rceil + 1 & if \quad n = 4k + 2 \ or \ 4k + 3. \end{cases}$

for $k > 1$.

Proof. Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$.

Let $k > 1$.

Case (i): $n = 4k$.

Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_{4k-1}, v_{4k}\}$.

Now, choose the first two of every four vertices of $V(C_n)$ starting from v_1 to get a minimum total restrained detour dominating set.

Hence, $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\}$ is a minimum total restrained detour dominating set of C_n.

Therefore, $\gamma_{trd}(C_n) = 2k$

$= \frac{n}{2}$

$= \left\lceil \frac{n}{2} \right\rceil$ where $n = 4k$.

Case (ii): $n = 4k + 1$.

Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_{4k}, v_{4k+1}\}$.

Proceeding as before $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\}$ is a subset of every total restrained detour dominating set of C_n.

But, S does not dominate all the vertices of C_n (S does not dominate v_{4k}).

Therefore, $S \cup \{v_{4k+1}\}$ is total restrained detour dominating sets of C_n with minimum cardinality.

Hence, $\gamma_{trd}(C_n) = |S| + 1$

$= 2k + 1$

$= \left\lceil \frac{n}{2} \right\rceil$ where $n = 4k + 1$.

Case (iii): $n = 4k + 2$.

Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_{4k+1}, v_{4k+2}\}$

Proceeding as before $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\}$ is a subset of every total restrained detour dominating set of C_n.

But, S does not dominate all the vertices of C_n (S does not dominate v_{4k} and v_{4k+1}) and $S \cup \{v_{4k+1}, v_{4k+2}\}$, is a minimum total restrained detour dominating sets of C_n.

Therefore, $\gamma_{trd}(C_n) = |S| + 2$

$= 2k + 2$

$= (2k + 1) + 1$

$= \left\lceil \frac{n}{2} \right\rceil + 1$ where $n = 4k + 2$.

Case (iv): $n = 4k + 3$.

Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_{4k+2}, v_{4k+3}\}$

Proceeding as before $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\}$ is a subset of every total restrained detour dominating set of C_n.

In this case, to get a minimum total restrained detour dominating set of C_n, we need a minimum of three more vertices, since no set of $|S| + 2$ vertices is satisfying the conditions.

Further, $S \cup \{v_{4k+1}, v_{4k+2}, v_{4k+3}\}$, is one of the minimum total restrained detour dominating sets of C_n.

Hence, $\gamma_{trd}(C_n) = |S| + 3$

$= 2k + 3$

$= (2k + 2) + 1$

$= \left\lceil \frac{n}{2} \right\rceil + 1$ where $n = 4k + 3$.

Theorem 2.8. $\gamma_{trd}(K_{1,n}) = n + 1$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \dots, v_n\}$

$S = \{v_1, v_2, \dots, v_n\}$ being the set of end vertices contained in every total restrained detour dominating set of $K_{1,n}$.

Further, $G[S]$ has an isolated vertex v.

Therefore, $S \cup \{v\}$ is a total restrained detour dominating set of $K_{1,n}$.

Hence, $\gamma_{trd}(K_{1,n}) = |S| + 1 = n + 1$.

Theorem 2.9. $\gamma_{trd}(K_n) = 2, \ n > 3$.

Proof. Any minimum detour dominating set of $K_n$contains exactly two vertices.

Being vertices of K_n, they are adjacent.

Further, $G[S]$ has no isolated vertices for all n.

Therefore, every two element subset of $V(K_n)$ is a minimum total restrained detour dominating sets.

Hence, $\gamma_{trd}(K_n) = \gamma_d(K_n) = 2$.

Theorem 2.10. $\gamma_{trd}(K_{m,n}) = 2$.

Proof. Let $V(K_{m,n}) = \{v_1, v_2, v_3, \dots, v_m, u_1, u_2, u_3, \dots, u_n\}$ and $|V_1| = m;$

$|V_2| = n$. Any two element set containing two vertices one vertex from V_1 and another vertex from V_2 and viceversa forms a minimum total restrained detour dominating set.

Also, the subgraph induced by S has no isolated vertex.

Therefore, $\gamma_{trd}(K_{m,n}) = 2$.

Theorem 2.11 $\gamma_{trd}(W_{1,p-1}) = 2$.

Proof. Let $V(W_{1,p-1}) = \{v, v_1, v_2, v_3, \ldots, v_{p-1}\}$.

For $i = 1\ to\ p-1$, $S_i = \{v, v_i\}$ forms a total restrained detour dominating set of $W_{1,p-1}$.

Also, the subgraph induced by S_i has no isolated vertex.

Therefore, S_i is a minimum total restrained detour dominating set of G.

Hence, $\gamma_{trd}(W_{1,p-1}) = |S_i| = 2$.

Theorem 2.12. If G ia caterpillar with p vertices such that for every cut vertex v of G there exists a pendant edge, then $\gamma_{trd}(G) = p$.

Proof. Since for every cut vertex v of G, there exists a vertex u such that uv is a pendant edge of G by 2.4, $\{u, v\}$ is a subset of every total restrained detour dominating set of G for every $v \in G$. Since in a caterpillar, every vertex is either a cut vertex or pendant vertex, $\gamma_{trd}(G) = p$.

Theorem 2.13. For every pair k, p of integers such that $2 \le k < p$, there exists a connected graph G of order p such that $\gamma_{trd}(G) = k$.

Proof. $If\ k = 2$, then $G \cong K_p$ satisfies the theorem.

Let $2 < k < p$ and $p \neq k + 1$, Let $C_5: v_1, v_2, v_3, v_4, v_5$ be a cycle of order 5 and let $P_2: u_1, u_2$ be a path of order 2. Let H be the graph obtained from C_5 and P_2 by the identifying the vertex v_4 in C_5 and u_1 in P_2. Attach new vertices $z_1, z_2, \ldots, z_{k-3}$ to u_2 and obtain the graph G as shown in figure 2.3.

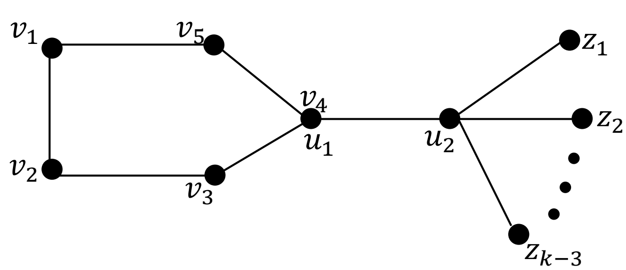

Figure 2.3

Let $S = \{z_1, z_2, \ldots, z_{k-3}\}$ be the set of all end vertices of G. Since S is a subset of every total restrained detour dominating set of G, $\gamma_{trd}(G) \ge k - 3$.

Now, $S' = S \cup \{v_1, v_2, u_2\}$ is a total detour dominating set of G and is minimum.

Further, $G[V - S']$ has no isolated vertices.

Therefore, S' is a minimum total restrained detour dominating set of G.

Hence, $\gamma_{trd}(G) = |S'| = k$.

Theorem 2.14. Let a and b be two positive integers with $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_d(G) = a$ and $\gamma_{trd}(G) = b$.

Proof. Case 1. $a = b = 2$.

Let $G \cong K_p$ be the complete graph. Then, $\gamma_d(G) = \gamma_{trd}(G) = 2$.

Case 2. $a = b > 2$.

Let $C_4: u_1, u_2, u_3, u_4, u_5, u_6$ be a cycle of order 6 and let $P_2: v_1, v_2$ be a path of order 2. Let H be the graph obtained from C_6 and P_2 by the identifying the vertex u_1 in C_6 and v_2 in P_2. Attach new vertices $w_1, w_2, \dots, w_{a-3}$ to u_4 and obtain the graph G as shown in figure 2.4.

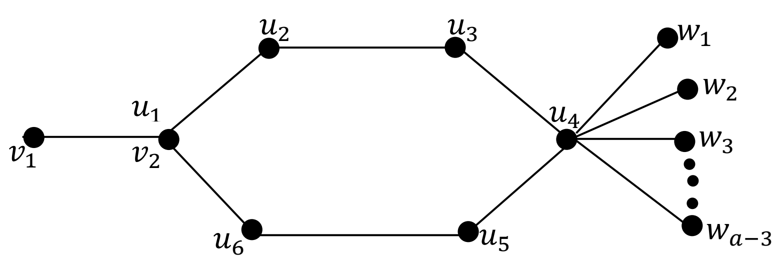

Figure 2.4

Let $S = \{v_1, w_1, w_2, \dots, w_{a-3}\}$.

Now, $S_1 = S \cup \{v_2, u_4\}$ is a total detour dominating set of G and is minimum.

Further, $G[V - S_1]$ has no isolated vertices.

Therefore, S_1 is a minimum detour dominating set and total restrained detour dominating set of G.

Hence, $\gamma_{trd}(G) = |S_1| = a = \gamma_d(G)$.

Case 3. $2 \leq a < b$.

Subcase 3a. $a = 2$ and $b = 3$.

Let $G \cong K_3$ be the complete graph. Then by Theorem 1.8, $\gamma_d(G) = 2$. Hence, $\gamma_{trd}(G) = 3$.

Subcase 3b. $2 < a < b$ and $b = a + 1$.

Let $C_4: u_1, u_2, u_3, u_4$ be a cycle of order 4 and let $P_3: w_1, w_2, w_3$ be a path of order 3. Let H be the graph obtained from C_4 and P_2 by the identifying the vertex u_4 in C_4. Now join u_1 and u_3. Attach new vertices $v_1, v_2, \dots, v_{a-2}$ to u_1 and obtain the graph G as shown in figure 2.5.

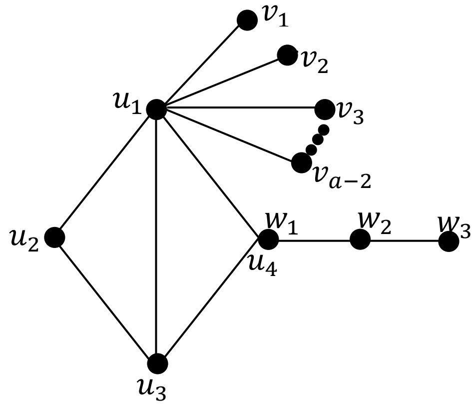

Figure 2.5

Let $S = \{w_3, v_1, v_2, \dots, v_{a-2}\}$.

Now, $S_1 = S \cup \{u_1\}$ is a minimum detour dominating set of G.

Hence, $\gamma_d(G) = |S_1| = a$.

Next, we have to show that $\gamma_{trd}(G) = a + 1$.

Since the subgraph $G[S_1]$ has an isolated vertex w_3, S_1 is not a minimum total restrained detour dominating set of G.

Now, $S_2 = S_1 \cup \{w_2\}$ is a minimum total restrained detour dominating set of G.

Hence, $\gamma_{trd}(G) = |S_1| = a + 1$.

Subcase 3c. $2 < a < b$ and $b \neq a + 1$.

Let $P: u_1, u_2$ be a path on two vertices. Let G be the graph in figure 2.6 obtained from P by adding $b - 2$ new vertices $z_1, z_2, \dots, z_{a-1}, v_1, v_2, v_3, \dots, v_{b-a-1}$ and joining each $z_i (1 \le i \le a - 1)$ with u_2 and each $v_i (1 \le i \le b - a - 1)$ with u_1 and u_2 and obtain the graph G as shown in figure 2.6.

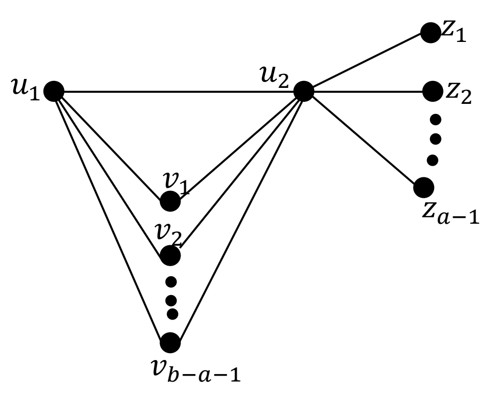

Figure 2.6

Let $Z = \{z_1, z_2, \dots, z_{a-1}\}$.

Now, $S = Z \cup \{u_1\}$ is a minimum detour dominating set of G.

Hence, $\gamma_d(G) = |S| = a$.

Next, we have to show that $\gamma_{trd}(G) = b$.

Now, $S' = S \cup \{u_2, v_1, v_2, v_3, \dots, v_{b-a-1}\}$ is a minimum total restrained detour dominating set of G.

Hence, $\gamma_{trd}(G) = |S'| = b$.

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