THEORY OF SIX - DIMENSIONAL FINSLER SPACES IN TERMS OF SCALARS

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ABSTRACT

It has been observed that there are two known methods to study higher-dimensional spaces. First method is of studying n-dimensional spaces and then putting the value of n, depending on the requirement. The second method is to initially assume the dimension depending on our requirement. The first method is good for abstraction, while the second method is good for practical purposes, but in second method it is very difficult to understand fundamentals related with the dimensions of the space and perform calculations. The present paper refers to the study of six-dimensional Finsler spaces in terms of scalars. Here we are following the second method and accordingly we have defined and studied six- unit vectors and their h-covariant derivatives. We have obtained expressions for tensors C_{ijk} and P_{ijk}. Several new tensors based on six-dimensional Finsler spaces related with tensors C_{ijk} and P_{ijk} have also been defined and studied.ss

KEYWORDS. Torsion Tensors, Cartan's C-tensors and P-tensors.

1. INTRODUCTION.

Two-dimensional Finsler spaces based on orthonormal frame consisting of normalised support element l^i and the unit vector m^i normal to l^i, were introduced by Berwald [1,2]. This study in three-dimensional Finsler spaces was continued by Moor [10], which consisted l^i, m^i and a vector n^i orthogonal to both l^i and m^i. Various aspects of such spaces have been studied by Nobuchara and Nagai [11], Matsumoto [7,8,9] Rastogi [14] and many others. Similar to three-dimensional Finsler spaces Rastogi [15], Pandey and Dwivedi [12] developed four-dimensional Finsler spaces. Recently Pandey, Dwivedi and Gupta [13] initiated study of five-dimensional Finsler spaces in terms of scalars, which has been further developed by Dwivedi, Rastogi and Dwivedi [4, 5].

The purpose of the present paper is to provide systematic development of the study of six-dimensional Finsler spaces based on the theory of normal frame fields.

2. UNIT VECTOR IN SIX-DIMENSIONAL FINSLER SPACES

Let F^6, be the six-dimensional Finsler space equipped with a fundamental function L(x,y). Let $\delta^{1\,2\,3\,4\,5\,6}_{i\,j\,k\,l\,m\,n}$ be the generalised Kronecker delta and let us put

$$\gamma_{ijklmn} = \delta^{1\,2\,3\,4\,5\,6}_{i\,j\,k\,l\,m\,n} \text{ and } \gamma^{ijklmn} = \delta^{i\,j\,k\,l\,m\,n}_{1\,2\,3\,4\,5\,6} \quad (2.1)$$

then the E-tensor defined by

$$\varepsilon_{ijklmn} = |g|^{1/2}\,\gamma_{ijklmn} \text{ and } \varepsilon^{ijklmn} = |g|^{-1/2}\,\gamma^{ijklmn} \quad (2.2)$$

where $|g|$ is the determinant consisting of the components of the fundamental tensor $g = (g_{ij})$.

In a six-dimensional Finsler space F^6, we have six-unit vectors defined as follows:

$$e_{1)l} = L^{-1}(1/2)(\Delta_i L^2) = l_i, \quad (2.3)$$

where $\Delta_i = \partial/\partial y^i$.

Corresponding to $e_{1)l}$, we put

$$N_{1)ij} = g_{ij} - e_{1)l}\, e_{1)j} = g_{ij} - l_i\, l_j = h_{ij} \quad (2.4)$$

Here the matrix $N_1 = (N_{1)ij})$ is of rank 5 and the vector $e_{1)l} = l_i$ is an eigen vector. Eigen values of $N_{1)}$ are all equal to 1. Also $E_{1)l} = N^i_{0)j}\, L^j_{1)}$, $e_{1)}{}^l = (E_1)^{-1}\, E_{1)}{}^i$.

In F^6, we introduce the second unit vector as follows:

$$e_{2)}{}^l = (L_2)^{-1}\, L_{2)}{}^l = (E_{2)})^{-1}\, E_{2)}{}^l \quad (2.5)$$

where L_2 is the length of $L_{2)}{}^l$, such that

$$L_{2)l} = (1/2)^2(\Delta_i\, \Delta_j\, \Delta_k\, L^2)\, g^{jk} = C_{ijk}\, g^{jk} = C_i \quad (2.6)$$

Hence $e_{2)}{}^l = C^{-1}\, C^i = m^i$. Corresponding to $e_{2)}{}^l$, we put

$$N_{2)ij} = N_{1)ij} - e_{2)l}\, e_{2)j} = h_{ij} - m_i\, m_j \quad (2.7)$$

Also $E_{3)}{}^i = N_{2)}{}^i{}_j\, L_{3)}{}^j$ and the matrix $N_{2)} = (N_{2)ij})$ is of rank 4, $e_{1)l}$ and e_{2i} are eigen vectors corresponding to eigen value zero and $N_{2)}$ and other eigen values are all equal to one. Thus $E_{3)}$ is orthogonal to $e_{1)}$ and $e_{2)}$ and the third unit vector is defined as

$$e_{3)}{}^l = (E_3)^{-1}\, E_{3)}{}^l = n_{(1)}{}^l \quad (2.8)$$

where E_3 is the length of $E_{3)}{}^i$.

Similar to above construction, we define

$$e_{4)}{}^{l} = (E_4)^{-1}\ E_{4)}{}^{l} = n_{(2)}{}^{l} \quad (2.9)$$

where $n_{(2)}{}^{l}$ is a unit vector orthogonal to l^i, m^i and $n_{(1)}{}^{i}$.

Also, we define

$$N_{3)ij} = N_{2)ij} - e_{3)l}\ e_{3)j} = h_{ij} - m_i\ m_j - n_{(1)l}\ n_{(1)j} \quad (2.10)$$

Further, we define

$$e_{5)}{}^{l} = (E_5)^{-1}\ E_{5)}{}^{l} = n_{(3)}{}^{l} \quad (2.11)$$

where $n_{(3)}{}^{l}$ is a unit vector orthogonal to l^i, m^i, $n_{(1)}{}^{l}$ and $n_{(2)}{}^{i}$.

We also define

$$N_{4)ij} = N_{3)ij} - e_{4)l}\ e_{4)j} = h_{ij} - m_i\ m_j - n_{(1)l}\ n_{(1)j} - n_{(2)l}\ n_{(2)j} \quad (2.12)$$

Lastly, we define

$$e_{6)}{}^{l} = (E_6)^{-1}\ E_{6)}{}^{l} = n_{(4)}{}^{l} \quad (2.13)$$

where $n_{(4)}{}^{l}$ is a unit vector orthogonal to l^i, m^i, $n_{(1)}{}^{l}$, $n_{(2)}{}^{l}$ and $n_{(3)}{}^{i}$. Also, we define

$$N_{5)ij} = N_{4)ij} - e_{5)l}\ e_{5)j} = h_{ij} - m_i\ m_j - n_{(1)l}\ n_{(1)j} - n_{(2)l}\ n_{(2)j} - n_{(3)l}\ n_{(3)j} \quad (2.14)$$

Thus $e_{\alpha)l}$ (α= 1,2,3,4,5,6), constitute an orthonormal frame corresponding to six-dimensional Finsler space F^6. Let T be a tensor of type (1,2) represented by T^i_{jk}, then the adopted scalar components $T_{\alpha\beta\gamma}$ (α,β,γ = 1,2,3,4,5,6) of T are defined by

$$T_{\alpha\beta\gamma} = T^i_{jk}\ e_{\alpha)l}\ e_{\beta)}{}^{j}\ e_{\gamma)}{}^{k} \quad (2.15)$$

Such that

$$T^i_{jk} = T_{\alpha\beta\gamma}\ e_{\alpha)}{}^{i}\ e_{\beta)j}\ e_{\gamma)k} \quad (2.16)$$

The adopted components of g_{ij} and ε_{ijklmn} are $\delta_{\alpha\beta}$ and $\varepsilon_{\alpha\beta\gamma\delta\Theta\phi} = (\delta^{1}{}_{\alpha}{}^{2}{}_{\beta}{}^{3}{}_{\gamma}{}^{4}{}_{\delta}{}^{5}{}_{\Theta}{}^{6}{}_{\phi})$.

3. h-COVARIANT DERIVATIVE

The h-covariant derivative $e_{\alpha)}{}^{i}{}_{/j}$ of the vector $e_{\alpha)}$ belonging to orthonormal frame is given by

$$e_{\alpha)}{}^{i}{}_{/j} = H_{\alpha)\beta\gamma}\, e_{\beta)}{}^{i}\, e_{\gamma)j} \qquad (3.1)$$

h-Connection Scalars. Let $H_{\alpha)\beta\gamma}$ be scalar components of the h-covariant derivative given by equation (3.1), then $H_{\alpha)\beta\gamma}$ are called h-connection scalars.

From equation (3.1), we can obtain

$$e_{1)}{}^{i}{}_{/j} = H_{1)\beta\gamma}\, e_{\beta)}{}^{i}\, e_{\gamma)j} = l^{i}{}_{/j} = 0 \qquad (3.2)$$

showing that $H_{1)\beta\gamma} = 0$. Also, the orthogonality of the frame field gives Matsumoto [8]:

$$H_{\alpha)\beta\gamma} = - H_{\beta)\alpha\gamma} = H_{\alpha)\alpha\gamma} = 0 \qquad (3.3)$$

Further we have defined and assumed h_j, k_j, r_j, s_j, t_j, u_j, v_j, w_j, x_j and y_j as ten h- connection vector, such that they are given by following equations:

$$H_{2)3\beta}\, e_{\beta)}{}^{j} = h_j = h_\beta\, e_{\beta)j},\ H_{3)4\beta}\, e_{\beta)}{}^{j} = k_j = k_\beta\, e_{\beta)j}, \qquad (3.4)\ a$$

$$H_{4)2\beta}\, e_{\beta)}{}^{j} = r_j = r_\beta\, e_{\beta)j},\ H_{5)2\beta}\, e_{\beta)}{}^{j} = s_j = s_\beta\, e_{\beta)j}, \qquad (3.4)\ b$$

$$H_{5)3\beta}\, e_{\beta)}{}^{j} = t_j = t_\beta\, e_{\beta)j},\ H_{5)4\beta}\, e_{\beta)}{}^{j} = u_j = u_\beta\, e_{\beta)j}, \qquad (3.4)\ c$$

$$H_{6)2\beta}\, e_{\beta)}{}^{j} = v_j = v_\beta\, e_{\beta)j},\ H_{6)3\beta}\, e_{\beta)}{}^{j} = w_j = w_\beta\, e_{\beta)j}, \qquad (3.4)\ d$$

$$H_{6)4\beta}\, e_{\beta)}{}^{j} = x_j = x_\beta\, e_{\beta)j},\ H_{6)5\beta}\, e_{\beta)}{}^{j} = y_j = y_\beta\, e_{\beta)j} \qquad (3.4)\ e$$

Using these equations we obtain an expression for $e_{2)}{}^{i}{}_{/j}$ in the following form:

$$e_{2)}{}^{i}{}_{/j} = H_{2)\beta\gamma}\, e_{\beta)}{}^{i}\, e_{\gamma)j} = H_{2)1\beta}\, e_{1)}{}^{i}\, e_{\beta)j} + H_{2)2\beta}\, e_{2)}{}^{i}\, e_{\beta)j} + H_{2)3\beta}\, e_{3)}{}^{i}\, e_{\beta)j}$$

$$+ H_{2)4\beta}\, e_{4)}{}^{i}\, e_{\beta)j} + H_{2)5\beta}\, e_{5)}{}^{l}\, e_{\beta)j} + H_{2)6\beta}\, e_{6)}{}^{l}\, e_{\beta)j},$$

which by virtue of (3.3) can be expressed as

$$e_{2)}{}^{i}{}_{/j} = m^{i}{}_{/j} = n_{(1)}{}^{l}\, h_j - n_{(2)}{}^{l}\, r_j - n_{(3)}{}^{l}\, s_j - n_{(4)}{}^{l}\, v_j \qquad (3.5)\ a$$

Similar to (3.5) a, we can obtain

$$e_{3)}{}^{i}{}_{/j} = n_{(1)}{}^{i}{}_{/j} = - m^{i}\, h_j + n_{(2)}{}^{l}\, k_j - n_{(3)}{}^{l}\, t_j - n_{(4)}{}^{l}\, w_j \qquad (3.5)\ b$$

$$e_{4)}{}^{i}{}_{/j} = n_{(2)}{}^{i}{}_{/j} = m^{i}\, r_j - n_{(1)}{}^{l}\, k_j - n_{(3)}{}^{l}\, u_j - n_{(4)}{}^{l}\, x_j \qquad (3.5)\ c$$

$$e_{5)}{}^{i}{}_{/j} = n_{(3)}{}^{i}{}_{/j} = m^{i}\, s_j + n_{(1)}{}^{l}\, t_j + n_{(2)}{}^{l}\, u_j - n_{(4)}{}^{l}\, y_j \qquad (3.5)\ d$$

$$e_{6)}{}^{i}{}_{/j} = n_{(4)}{}^{i}{}_{/j} = m^{i}\, v_j + n_{(1)}{}^{l}\, w_j + n_{(2)}{}^{l}\, x_j + n_{(3)}{}^{l}\, y_j \qquad (3.5)\ e$$

The matrix related with h-connection scalars is expressed as($H_{\alpha)\beta\gamma}$).

4. TORSION TENSOR C_{ijk}

Any third order symmetric tensor C_{ijk} satisfying $C_{ijk}\, l^i = 0$, in a six-dimensional Finsler space F^6, can be expressed as

$$
\begin{aligned}
L\, C_{ijk} = {} & C_{(1)}\, m_i\, m_j\, m_k + C_{(2)}\, n_{(1)l}\, n_{(1)j}\, n_{(1)k} + C_{(3)}\, n_{(2)l}\, n_{(2)j}\, n_{(2)k} + C_{(4)}\, n_{(3)l}\, n_{(3)j}\, n_{(3)k} \\
& + C_{(5)}\, n_{(4)l}\, n_{(4)j}\, n_{(4)k} + \textstyle\sum_{(l,j,k)}[C_{(6)}\, m_i\, m_j\, n_{(1)k} + C_{(7)}\, m_i\, m_j\, n_{(2)k} \\
& + C_{(8)}\, m_i\, m_j\, n_{(3)k} + C_{(9)} m_i\, m_j\, n_{(4)k} + C_{(10)}\, n_{(1)l}\, n_{(1)j}\, m_k \\
& + C_{(11)}\, n_{(1)l}\, n_{(1)j}\, n_{(2)k} + C_{(12)}\, n_{(1)l}\, n_{(1)j}\, n_{(3)k} + C_{(13)}\, n_{(1)l}\, n_{(1)j}\, n_{(4)k} \\
& + C_{(14)}\, n_{(2)l}\, n_{(2)j}\, m_k + C_{(15)}\, n_{(2)l}\, n_{(2)j}\, n_{(1)k} + C_{(16)}\, n_{(2)l}\, n_{(2)j}\, n_{(3)k} \\
& + C_{(17)}\, n_{(2)l}\, n_{(2)j}\, n_{(4)k} + C_{(18)}\, n_{(3)l}\, n_{(3)j}\, m_k + C_{(19)}\, n_{(3)l}\, n_{(3)j}\, n_{(1)k} \\
& + C_{(20)}\, n_{(3)l}\, n_{(3)j}\, n_{(2)k} + C_{(21)}\, n_{(3)l}\, n_{(3)j}\, n_{(4)k} + C_{(22)}\, n_{(4)l}\, n_{(4)j}\, m_k \\
& + C_{(23)}\, n_{(4)l}\, n_{(4)j}\, n_{(1)k} + C_{(24)}\, n_{(4)l}\, n_{(4)j}\, n_{(2)k} + C_{(25)}\, n_{(4)l}\, n_{(4)j}\, n_{(3)k} \\
& + C_{(26)}\, m_i\, (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j}) + C_{(27)}\, m_i (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) \\
& + C_{(28)}\, m_i\, (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + C_{(29)}\, m_i\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) \\
& + C_{(30)}\, m_i\, (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) + C_{(31)}\, m_i\, (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) \\
& + C_{(32)}\, n_{(1)l}\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) + C_{(33)}\, n_{(1)i} (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) \\
& + C_{(34)}\, n_{(1)l} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) + C_{(35)}\, n_{(2)i} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j})] \qquad (4.1)
\end{aligned}
$$

Multiplying equation (4.1) by g^{jk}, we obtain

$$
\begin{aligned}
LC_i = {} & C_{(1)}\, m_i + C_{(2)}\, n_{(1)l} + C_{(3)}\, n_{(2)l} + C_{(4)}\, n_{(3)l} + C_{(5)}\, n_{(4)l} + C_{(6)}\, n_{(1)l} + C_{(7)}\, n_{(2)i} \\
& + C_{(8)}\, n_{(3)l} + C_{(9)}\, n_{(4)l} + C_{(10)}\, m_i + C_{(11)}\, n_{(2)l} + C_{(12)}\, n_{(3)l} + C_{(13)}\, n_{(4)i} \\
& + C_{(14)}\, m_i + C_{(15)}\, n_{(1)l} + C_{(16)}\, n_{(3)l} + C_{(17)}\, n_{(4)l} + C_{(18)}\, m_i + C_{(19)}\, n_{(1)i} \\
& + C_{(20)}\, n_{(2)l} + C_{(21)}\, n_{(4)l} + C_{(22)}\, m_i + C_{(23)}\, n_{(1)l} + C_{(24)}\, n_{(2)l} + C_{(25)}\, n_{(3)l} \qquad (4.2)
\end{aligned}
$$

which leads to

$$C_{(1)} + C_{(10)} + C_{(14)} + C_{(18)} + C_{(22)} = LC,\; C_{(2)} + C_{(6)} + C_{(15)} + C_{(19)} + C_{(23)} = 0, \qquad (4.3)\text{ a}$$

$$C_{(3)} + C_{(7)} + C_{(11)} + C_{(20)} + C_{(24)} = 0,\; C_{(4)} + C_{(8)} + C_{(12)} + C_{(16)} + C_{(25)} = 0, \qquad (4.3)\text{ b}$$

$$C_{(5)} + C_{(9)} + C_{(13)} + C_{(17)} + C_{(21)} = 0. \qquad (4.3)\text{ c}$$

Also from equation (4.1), we can observe that $C_{(26)}$, $C_{(27)}$, $C_{(28)}$, $C_{(29)}$, $C_{(30)}$,

$C_{(31)}$, $C_{(32)}$, $C_{(33)}$, $C_{(34)}$ and $C_{(35)}$ are ten non-zero scalars in F^6.

Alternatively, using $C_{\alpha\beta\gamma} = L\, C_{ijk}\, e_{\alpha)}{}^{I}\, e_{\beta)}{}^{j}\, e_{\gamma)}{}^{k}$, we can observe that

$C_{1\beta\gamma} = L\, C_{ijk}\, e_{1)}{}^{I}\, e_{\beta)}{}^{j}\, e_{\gamma)}{}^{k} = L\, C_{ijk}\, l^{i}\, e_{\beta)}{}^{j}\, e_{\gamma)}{}^{k} = 0$, $C_{2\beta\gamma} = L\, C_{ijk}\, m^{i}\, e_{\beta)}{}^{j}\, e_{\gamma)}{}^{k}$, (4.4) a

$C_{2\beta\beta} = L\, C_{ijk}\, m^{i}\, e_{\beta)}{}^{j}\, e_{\beta)}{}^{k} = LC$, $C_{3\beta\beta} = 0 = C_{4\beta\beta} = C_{5\beta\beta} = C_{6\beta\beta}$, (4.4) b

$C_{222} + C_{233} + C_{244} + C_{255} + C_{266} = LC$, $C_{322} + C_{333} + C_{344} + C_{355} + C_{366} = 0$, (4.4) c

$C_{422} + C_{433} + C_{444} + C_{455} + C_{466} = 0$, $C_{522} + C_{533} + C_{544} + C_{555} + C_{566} = 0$, (4.4) d

$C_{622} + C_{633} + C_{644} + C_{655} + C_{666} = 0$. (4.4)e

Further, we can observe that 10 non-zero scalars given above can also be expressed as

$C_{234} = L\, C_{ijk}\, m^{i}\, n_{(1)}{}^{j}\, n_{(2)}{}^{k}$, $C_{235} = L\, C_{ijk}\, m^{i}\, n_{(1)}{}^{j}\, n_{(2)}{}^{k}$, $C_{236} = L\, C_{ijk}\, m^{i}\, n_{(1)}{}^{j}\, n_{(3)}{}^{k}$, (4.5) a

$C_{245} = L\, C_{ijk}\, m^{i}\, n_{(2)}{}^{j}\, n_{(3)}{}^{k}$, $C_{246} = L\, C_{ijk}\, m^{i}\, n_{(2)}{}^{j}\, n_{(4)}{}^{k}$, $C_{256} = L\, C_{ijk}\, m^{i}\, n_{(3)}{}^{j}\, n_{(4)}{}^{k}$, (4.5) b

$C_{345} = L\, C_{ijk}\, n_{(1)}{}^{I}\, n_{(2)}{}^{j}\, n_{(3)}{}^{k}$, $C_{346} = L\, C_{ijk}\, n_{(1)}{}^{I}\, n_{(2)}{}^{j}\, n_{(4)}{}^{k}$, (4.5) c

$C_{356} = L\, C_{ijk}\, n_{(1)}{}^{I}\, n_{(3)}{}^{j}\, n_{(4)}{}^{k}$, $C_{456} = L\, C_{ijk}\, n_{(2)}{}^{I}\, n_{(3)}{}^{j}\, n_{(4)}{}^{k}$ (4.5) d

Hence:

Theorem 4.1.: In a six-dimensional Finsler space F^6, Cartan's torsion tensor C_{ijk} has 25 coefficients represented by equations (4.3) a, (4.3) b and (4.3) c or (4.4)c, (4.4) d and (4.4) e and ten non-zero scalars $C_{(26)}$, $C_{(27)}$, $C_{(28)}$, $C_{(29)}$, $C_{(30)}$, $C_{(31)}$, $C_{(32)}$, $C_{(33)}$, $C_{(34)}$ and $C_{(35)}$ or scalars represented by equations (4.5) a, (4.5) b, (4.5) c and (4.5)d.

It has been observed that it is possible to predict number of terms in the expression for C_{ijk} in an n-dimensional Finsler space. For this we propose:

Theorem 4.2.: In an n-dimensional Finsler space, the number of terms in the expression for C_{ijk} is given by (1/6) (n-1) n (n+1) and the number of non-zero scalars for $n \geq 4$, is given by (1/6)(n-1)(n-2)(n-3).

Proof.: It can be observed that for n = 2, the number of terms in C_{ijk} is 1, for n = 3, it is 4, for n = 4, it is 10, for n = 5, it is 20 and for n = 6 it is 35. Also, the number of non-zero scalars for n = 4 is 1, for n = 5 is 4 and for n = 6 is 10 etc. Hence this theorem can be proved easily by method of induction.

In the next section we shall be studying tensors similar to C-tensors defined and studied by Shimada [18] and Rastogi [16].

5. CARTAN'S C-TENSORS IN F^6

From equation (4.1) by virtue of ${}^1C_{jk} = L\, C_{ijk}\, m^i$, ${}^2C_{jk} = L\, C_{ijk}\, n_{(1)}^{I}$, ${}^3C_{jk} = L\, C_{ijk}\, n_{(2)}^{I}$, ${}^4C_{jk} = L\, C_{ijk}\, n_{(3)}^{I}$ and ${}^5C_{jk} = L\, C_{ijk}\, n_{(4)}^{I}$, we can define following symmetric C-tensors:

$$\begin{aligned} {}^1C_{jk} = {} & C_{(1)}\, m_j\, m_k + C_{(6)} (m_j\, n_{(1)k} + m_k\, n_{(1)j}) + C_{(7)} (m_j\, n_{(2)k} + m_k\, n_{(2)j}) \\ & + C_{(8)} (m_j\, n_{(3)k} + m_k\, n_{(3)j}) + C_{(9)} (m_j\, n_{(4)k} + m_k\, n_{(4)j}) \\ & + C_{(10)}\, n_{(1)j}\, n_{(1)k} + C_{(14)}\, n_{(2)j}\, n_{(2)k} + C_{(18)}\, n_{(3)j}\, n_{(3)k} + C_{(22)}\, n_{(4)j}\, n_{(4)k} \\ & + C_{(26)} (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j}) + C_{(27)} (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) \\ & + C_{(28)} (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + C_{(29)} (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) \\ & + C_{(30)} (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) + C_{(31)} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) \end{aligned} \tag{5.1}$$

$$\begin{aligned} {}^2C_{jk} = {} & C_{(2)}\, n_{(1)j}\, n_{(1)k} + C_{(6)}\, m_j\, m_k + C_{(10)} (m_j\, n_{(1)k} + m_K\, n_{(1)j}) + C_{(11)} (n_{(1)j}\, n_{(2)k} \\ & + n_{(1)k}\, n_{(2)j}) + C_{(12)} (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) + C_{(13)} (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) \\ & + C_{(15)}\, n_{(2)j}\, n_{(2)k} + C_{(19)}\, n_{(3)j}\, n_{(3)k} + C_{(23)}\, n_{(4)j}\, n_{(4)k} + C_{(26)} (m_j\, n_{(2)k} + m_k\, n_{(2)j}) \\ & + C_{(27)} (m_j\, n_{(3)k} + m_k\, n_{(3)j}) + C_{(28)} (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + C_{(32)} (n_{(2)j}\, n_{(3)k} \\ & + n_{(2)k}\, n_{(3)j}) + C_{(33)} (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) + C_{(34)} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) \end{aligned} \tag{5.2}$$

$$\begin{aligned} {}^3C_{jk} = {} & C_{(3)}\, n_{(2)j}\, n_{(2)k} + C_{(7)}\, m_j\, m_k + C_{(11)}\, n_{(1)j}\, n_{(1)k} + C_{(14)} (m_j\, n_{(2)k} + m_k\, n_{(2)j}) \\ & + C_{(15)} (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j}) + C_{(16)} (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) + C_{(17)} (n_{(2)j}\, n_{(4)k} \\ & + n_{(2)k}\, n_{(4)j}) + C_{(20)}\, n_{(3)j}\, n_{(3)k} + C_{(24)}\, n_{(4)j}\, n_{(4)k} + C_{(26)} (m_j\, n_{(1)k} + m_k\, n_{(1)j}) \\ & + C_{(29)} (m_j\, n_{(3)k} + m_k\, n_{(3)j}) + C_{(30)} (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + C_{(32)} (n_{(1)j}\, n_{(3)k} \\ & + n_{(1)k}\, n_{(3)j}) + C_{(33)} (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + C_{(35)} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) \end{aligned} \tag{5.3}$$

$$\begin{aligned} {}^4C_{jk} = {} & C_{(4)}\, n_{(3)j}\, n_{(3)k} + C_{(8)}\, m_j\, m_k + C_{(12)}\, n_{(1)j}\, n_{(1)k} + C_{(16)}\, n_{(2)j}\, n_{(2)k} + C_{(18)} (m_j\, n_{(3)k} \\ & + m_k\, n_{(3)j}) + C_{(19)} (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) + C_{(20)} (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) \\ & + C_{(21)} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) + C_{(25)}\, n_{(4)j}\, n_{(4)k} + C_{(27)} (m_j\, n_{(1)k} + m_k\, n_{(1)j}) \\ & + C_{(29)} (m_j\, n_{(2)k} + m_k\, n_{(2)j}) + C_{(31)} (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + C_{(32)} (n_{(1)j}\, n_{(2)k} \\ & + n_{(1)k}\, n_{(2)j}) + C_{(34)} (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + C_{(35)} (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) \end{aligned} \tag{5.4}$$

and

$$^5C_{jk} = C_{(5)}\, n_{(4)j}\, n_{(4)k} + C_{(9)}\, m_j\, m_k + C_{(13)}\, n_{(1)j}\, n_{(1)k} + C_{(17)}\, n_{(2)j}\, n_{(2)k} + C_{(21)}\, n_{(3)j}\, n_{(3)k}$$
$$+ C_{(22)}\, (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + C_{(23)}\, (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + C_{(24)}\, (n_{(2)j}\, n_{(4)k}$$
$$+ n_{(2)k}\, n_{(4)j}) + C_{(25)}\, (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) + C_{(28)}\, (m_j\, n_{(1)k} + m_k\, n_{(1)j})$$
$$+ C_{(30)}\, (m_j\, n_{(2)k} + m_k\, n_{(2)j}) + C_{(31)}\, (m_j\, n_{(3)k} + m_k\, n_{(3)j}) + C_{(33)}\, (n_{(1)j}\, n_{(2)k}$$
$$+ n_{(1)k}\, n_{(2)j}) + C_{(34)}\, (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) + C_{(35)}\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) \quad (5.5)$$

From these equations we can obtain

$$^1C_{jk}\, m^k = C_{(1)}\, m_j + C_{(6)}\, n_{(1)j} + C_{(7)}\, n_{(2)j} + C_{(8)}\, n_{(3)j} + C_{(9)}\, n_{(4)j}, \quad (5.6)a$$

$$^2C_{jk}\, m^k = C_{(6)}\, m_j + C_{(10)}\, n_{(1)j} + C_{(26)}\, n_{(2)j} + C_{(27)}\, n_{(3)j} + C_{(28)}\, n_{(4)j}, \quad (5.6)b$$

$$^3C_{jk}\, m^k = C_{(7)}\, m_j + C_{(26)}\, n_{(1)j} + C_{(14)}\, n_{(2)j} + C_{(29)}\, n_{(3)j} + C_{(30)}\, n_{(4)j}, \quad (5.6)c$$

$$^4C_{jk}\, m^k = C_{(8)}\, m_j + C_{(27)}\, n_{(1)j} + C_{(29)}\, n_{(2)j} + C_{(18)}\, n_{(3)j} + C_{(31)}\, n_{(4)j}, \quad (5.6)d$$

$$^5C_{jk}\, m^k = C_{(9)}\, m_j + C_{(28)}\, n_{(1)j} + C_{(30)}\, n_{(2)j} + C_{(31)}\, n_{(3)j} + C_{(22)}\, n_{(4)j}, \quad (5.6)e$$

which together with equations (5.1), (5.2), (5.3), (5.4) and (5.5) help us in getting

$$^2C_{jk}\, m^k = {}^1C_{jk}\, n_{(1)}{}^k,\ {}^3C_{jk}\, m^k = {}^1C_{jk}\, n_{(2)}{}^k,\ {}^3C_{jk}\, n_{(1)}{}^k = {}^2C_{jk}\, n_{(2)}{}^k,\ {}^4C_{jk}\, m^k$$
$$= {}^1C_{jk}\, n_{(3)}{}^k,\ {}^4C_{jk}\, n_{(1)}{}^k = {}^2C_{jk}\, n_{(3)}{}^k,\ {}^4C_{jk}\, n_{(2)}{}^k = {}^3C_{jk}\, n_{(3)}{}^k,\ {}^5C_{jk}\, n_{(3)}{}^k = {}^4C_{jk}\, n_{(4)}{}^k,$$
$$^5C_{jk}\, n_{(2)}{}^k = {}^3C_{jk}\, n_{(4)}{}^k,\ {}^5C_{jk}\, m^k = {}^1C_{jk}\, n_{(4)}{}^k,\ {}^5C_{jk}\, n_{(1)}{}^k = {}^2C_{jk}\, n_{(4)}{}^k \quad (5.7)$$

Hence:

Theorem 5.1.: In a six -dimensional Finsler space Cartan's C-tensors satisfy equations given in (5.7).

Also, by using equations (4.3) a, b, c we can observe that

$$^1C_{jk}\, m^k + {}^2C_{jk}\, n_{(1)}{}^k + {}^3C_{jk}\, n_{(2)}{}^k + {}^4C_{jk}\, n_{(3)}{}^k + {}^5C_{jk}\, n_{(4)}{}^k = L\, C_j \quad (5.8)$$

Hence:

Theorem 5.2.: In a six-dimensional Finsler space Cartan's C-tensors satisfy equation (5.8).

6. TORSION TENSOR P_{ijk}

Using $P_{ijk} = L\, C_{ijk/0}$, with the help of equations (3.5) a, b, c, d, e and (4.1), after very lengthy calculation we can obtain

$$P_{ijk} = L[\, C_{(1)/0}\, m_i\, m_j\, m_k + C_{(2)/0}\, n_{(1)l}\, n_{(1)j}\, n_{(1)k} + C_{(3)/0}\, n_{(2)l}\, n_{(2)j}\, n_{(2)k}$$

$$+ C_{(4)/0}\, n_{(3)l}\, n_{(3)j}\, n_{(3)k} + C_{(5)/0}\, n_{(4)l}\, n_{(4)j}\, n_{(4)k} + \sum_{(l,j,k)} [\, m_i\, m_j\{C_{(1)}\, m_{k/0}$$

$$+ (C_{(6)}\, n_{(1)k})_{/0} + (C_{(7)}\, n_{(2)k})_{/0} + (C_{(8)}\, n_{(3)k})_{/0} + (C_{(9)}\, n_{(4)k})_{/0}\}$$

$$+ n_{(1)l}\, n_{(1)j}\{C_{(2)}\, n_{(1)k/0} + (C_{(10)}\, m_k)_{/0} + (C_{(11)}\, n_{(2)k})_{/0} + (C_{(12)}\, n_{(3)k})_{/0}$$

$$+ (C_{(13)}\, n_{(4)k})_{/0}\} + n_{(2)l}\, n_{(2)j}\{C_{(3)}\, n_{(2)k/0} + (C_{(14)}\, m_k)_{/0} + (C_{(15)}\, n_{(1)k})_{/0}$$

$$+ (C_{(16)}\, n_{(3)k})_{/0} + (C_{(17)}\, n_{(4)k})_{/0}\} + n_{(3)l}\, n_{(3)j}\{C_{(4)}\, n_{(3)k/0} + (C_{(18)}\, m_k)_{/0}$$

$$+ (C_{(19)}\, n_{(1)k})_{/0} + (C_{(20)}\, n_{(2)k})_{/0} + (C_{(21)}\, n_{(4)k})_{/0}\} + n_{(4)l}\, n_{(4)j}\{C_{(5)}\, n_{(4)k/0}$$

$$+ (C_{(22)}\, m_k)_{/0} + (C_{(23)}\, n_{(1)k})_{/0} + (C_{(24)}\, n_{(2)k})_{/0} + (C_{(25)}\, n_{(3)k})_{/0}\}$$

$$+ (m_i\, n_{(1)j} + m_j\, n_{(1)i})\{C_{(6)}\, m_{k/0} + C_{(10)}\, n_{(1)k/0} + (C_{(26)}\, n_{(2)k})_{/0}$$

$$+ (C_{(27)}\, n_{(3)k})_{/0} + (C_{(28)}\, n_{(4)k})_{/0}\} + (m_i\, n_{(2)j} + m_j\, n_{(2)i})\{C_{(7)}\, m_{k/0}$$

$$+ (C_{(26)}\, n_{(1)k})_{/0} + C_{(14)}\, n_{(2)k/0} + (C_{(29)}\, n_{(3)k})_{/0} + (C_{(30)}\, n_{(4)k})_{/0}\} + (m_i\, n_{(3)j}$$

$$+ m_j\, n_{(3)i})\{C_{(8)}\, m_{k/0} + (C_{(27)}\, n_{(1)k})_{/0} + (C_{(29)}\, n_{(2)k})_{/0} + C_{(18)}\, n_{(3)k/0}$$

$$+ (C_{(31)}\, n_{(4)k})_{/0}\} + (m_i\, n_{(4)j} + m_j\, n_{(4)i})\{C_{(9)}\, m_{k/0} + (C_{(28)}\, n_{(1)k})_{/0}$$

$$+ (C_{(30)}\, n_{(2)k})_{/0} + C_{(31)}\, n_{(3)k/0} + C_{(22)}\, n_{(4)k/0}\} + (n_{(1)l}\, n_{(2)j} + n_{(1)j}\, n_{(2)i})$$

$$\{(C_{(26)}\, m_k)_{/0} + C_{(11)}\, n_{(1)k/0} + C_{(15)}\, n_{(2)k/0} + (C_{(32)}\, n_{(3)k})_{/0} + (C_{(33)}\, n_{(4)k})_{/0}\}$$

$$+ (n_{(1)l}\, n_{(3)j} + n_{(1)j}\, n_{(3)i})\{(C_{(27)}\, m_k)_{/0} + C_{(12)}\, n_{(1)k/0} + (C_{(32)}\, n_{(2)k})_{/0}$$

$$+ C_{(19)}\, n_{(3)k/0} + (C_{(34)}\, n_{(4)k})_{/0}\} + (n_{(1)l}\, n_{(4)j} + n_{(1)j}\, n_{(4)i})\{(C_{(28)}\, m_k)_{/0}$$

$$+ C_{(13)}\, n_{(1)k/0} + (C_{(33)}\, n_{(2)k})_{/0} + (C_{(34)}\, n_{(3)k})_{/0} + C_{(23)}\, n_{(4)k/0}\} + (n_{(2)l}\, n_{(3)j}$$

$$+ n_{(2)j}\, n_{(3)i})\{(C_{(29)}\, m_k)_{/0} + (C_{(32)}\, n_{(1)k})_{/0} + C_{(16)}\, n_{(2)k/0} + C_{(20)}\, n_{(3)k/0}$$

$$+ (C_{(35)}\, n_{(4)k})_{/0}\} + (n_{(2)l}\, n_{(4)j} + n_{(2)j}\, n_{(4)i})\{(C_{(30)}\, m_k)_{/0} + (C_{(33)}\, n_{(1)k})_{/0}$$

$$+ C_{(17)}\, n_{(2)k/0} + (C_{(35)}\, n_{(3)k})_{/0} + C_{(24)}\, n_{(4)k/0}\} + (n_{(3)l}\, n_{(4)j} + n_{(3)j}\, n_{(4)i})$$

$$\{(C_{(31)}\, m_k)_{/0} + (C_{(34)}\, n_{(1)k})_{/0} + (C_{(35)}\, n_{(2)k})_{/0} + C_{(21)}\, n_{(3)k/0} + C_{(25)}\, n_{(4)k/0}\}]] \quad (6.1)$$

From equations (3.5) a, b, c, d, e, we can obtain

$m^i{}_{/0} = n_{(1)}{}^l h_0 - n_{(2)}{}^l r_0 - n_{(3)}{}^l s_0 - n_{(4)}{}^l v_0$ (6.2)a

$n_{(1)}{}^i{}_{/0} = - m^i h_0 + n_{(2)}{}^l k_0 - n_{(3)}{}^l t_0 - n_{(4)}{}^l w_0$ (6.2)b

$n_{(2)}{}^i{}_{/0} = m^i r_0 - n_{(1)}{}^l k_0 - n_{(3)}{}^l u_0 - n_{(4)}{}^l x_0$ (6.2)c

$n_{(3)}{}^i{}_{/0} = m^i s_0 + n_{(1)}{}^l t_0 + n_{(2)}{}^l u_0 - n_{(4)}{}^l y_0$ (6.2)d

$n_{(4)}{}^i{}_{/0} = m^i v_0 + n_{(1)}{}^l w_0 + n_{(2)}{}^l x_0 + n_{(3)}{}^l y_0$ (6.2)e.

Substituting in equation (6.1) the values of the h-covariant derivatives from equations (6.2) a, b, c, d, e of the five vectors $m_{i,}$ $n_{(1)l}$, $n_{(2)l}$, $n_{(3)l}$ and $n_{(4)l}$, we obtain the value of the torsion tensor P_{ijk} in the following form:

$P_{ijk} = L\, [A_{(1)}\, m_i\, m_j\, m_k + A_{(2)}\, n_{(1)l}\, n_{(1)j}\, n_{(1)k} + A_{(3)}\, n_{(2)l}\, n_{(2)j}\, n_{(2)k}$

$+ A_{(4)}\, n_{(3)l}\, n_{(3)j}\, n_{(3)k} + A_{(5)}\, n_{(4)l}\, n_{(4)j}\, n_{(4)k} + \sum_{(l,j,k)} [A_{(6)}\, m_i\, m_j\, n_{(1)k}$

$+ A_{(7)}\, m_i\, m_j\, n_{(2)k} + A_{(8)}\, m_i\, m_j\, n_{(3)k} + A_{(9)} m_i\, m_j\, n_{(4)k}$

$+ A_{(10)}\, n_{(1)l}\, n_{(1)j}\, m_k + A_{(11)}\, n_{(1)l}\, n_{(1)j}\, n_{(2)k} + A_{(12)}\, n_{(1)l}\, n_{(1)j}\, n_{(3)k}$

$+ A_{(13)}\, n_{(1)l}\, n_{(1)j}\, n_{(4)k} + A_{(14)}\, n_{(2)l}\, n_{(2)j}\, m_k + A_{(15)}\, n_{(2)l}\, n_{(2)j}\, n_{(1)k}$

$+ A_{(16)}\, n_{(2)l}\, n_{(2)j}\, n_{(3)k} + A_{(17)}\, n_{(2)l}\, n_{(2)j}\, n_{(4)k} + A_{(18)}\, n_{(3)l}\, n_{(3)j}\, m_k$

$+ A_{(19)}\, n_{(3)l}\, n_{(3)j}\, n_{(1)k} + A_{(20)}\, n_{(3)l}\, n_{(3)j}\, n_{(2)k} + A_{(21)}\, n_{(3)l}\, n_{(3)j}\, n_{(4)k}$

$+ A_{(22)}\, n_{(4)l}\, n_{(4)j}\, m_k + A_{(23)}\, n_{(4)l}\, n_{(4)j}\, n_{(1)k} + A_{(24)}\, n_{(4)l}\, n_{(4)j}\, n_{(2)k}$

$+ A_{(25)}\, nA_{(4)l}\, n_{(4)j}\, n_{(3)k} + A_{(26)}\, m_i\, (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})$

$+ A_{(27)}\, m_i (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) + A_{(28)}\, m_i\, (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j})$

$+ A_{(29)}\, m_i\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) + A_{(30)}\, m_i\, (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j})$

$+ A_{(31)}\, m_i\, (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) + A_{(32)}\, n_{(1)l}\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})$

$+ A_{(33)}\, n_{(1)i} (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) + A_{(34)}\, n_{(1)l} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j})$

$+ A_{(35)}\, n_{(2)i} (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j})]$ (6.3)

In equation (6.3), the terms involving coefficients $A_{(1)}$ to $A_{(35)}$ after a very rigorous calculation can be obtained in the following form:

$A_{(1)} = C_{(1)/0} - 3\, (C_{(6)}\, h_0 - C_{(7)}\, r_0 - C_{(8)}\, s_0 - C_{(9)}\, v_0)$,

$A_{(2)} = C_{(2)/0} + 3\, (C_{(10)}\, h_0 - C_{(11)}\, k_0 + C_{(12)}\, t_0 + C_{(13)}\, w_0)$,

$$A_{(3)} = C_{(3)/0} - 3\,(C_{(14)}\, r_0 - C_{(15)}\, k_0 - C_{(16)}\, u_0 - C_{(17)}\, x_0),$$

$$A_{(4)} = C_{(4)/0} - 3\,(C_{(18)}\, s_0 + C_{(19)}\, t_0 + C_{(20)}\, u_0 - C_{(21)}\, y_0),$$

$$A_{(5)} = C_{(5)/0} - 3(C_{(22)}\, v_0 + C_{(23)}\, w_0 + C_{(24)}\, x_0 + C_{(25)}\, y_0), \qquad (6.4)$$

$$A_{(6)} = C_{(6)/0} + (C_{(1)} - 2\, C_{(10)})\, h_0 - C_{(7)}\, k_0 + C_{(8)}\, t_0 + C_{(9)}\, w_0$$

$$+ 2(C_{(26)}\, r_0 + C_{(27)}\, s_0 + C_{(28)}\, v_0),$$

$$A_{(7)} = C_{(7)0} - (C_{(1)} - 2\, C_{(14)})\, r_0 + C_{(6)}\, k_0 + C_{(8)}\, u_0 + C_{(9)}\, x_0$$

$$-2(C_{(26)}\, h_0 - C_{(29)}\, s_0 - C_{(30)}\, v_0),$$

$$A_{(8)} = C_{(8)/0} - (C_{(1)} - 2\, C_{(18)})\, s_0 - C_{(6)}\, t_0 - C_{(7)}\, u_0 + C_{(9)}\, y_0$$

$$- 2(C_{(27)}\, h_0 - C_{(29)}\, r_0 - C_{(31)}\, v_0),$$

$$A_{(9)} = C_{(9)/0} - (C_{(1)} - 2\, C_{(22)})\, v_0 - C_{(6)}\, w_0 - C_{(7)}\, x_0 - C_{(8)}\, y_0$$

$$- 2(C_{(28)}\, h_0 - C_{(30)}\, r_0 - C_{(31)}\, s_0),$$

$$A_{(10)} = C_{(10)/0} - (C_{(2)} - 2\, C_{(6)})h_0 + C_{(11)}\, r_0 + C_{(12)}\, s_0 + C_{(13)}\, v_0$$

$$-2(C_{(26)}\, k_0 - C_{(27)}\, t_0 - C_{(28)}\, w_0),$$

$$A_{(11)} = C_{(11)/0} + (C_{(2)} - 2\, C_{(15)})k_0 - C_{(10)}\, r_0 + C_{(12)}\, u_0 + C_{(13)}\, x_0$$

$$+ 2(C_{(26)}\, h_0 + C_{(32)}\, t_0 + C_{(33)}\, w_0),$$

$$A_{(12)} = C_{(12)/0} - (C_{(2)} - 2\, C_{(19)})t_0 - C_{(10)}\, s_0 - C_{(11)}\, u_0 + C_{(13)}\, y_0$$

$$+ 2(C_{(27)}\, h_0 - C_{(32)}\, k_0 + C_{(34)}\, w_0),$$

$$A_{(13)} = C_{(13)/0} - (C_{(2)} - 2\, C_{(23)})w_0 - C_{(10)}\, v_0 - C_{(11)}\, x_0 - C_{(12)}\, y_0$$

$$+ 2(C_{(28)}\, h_0 - C_{(33)}\, k_0 + C_{(34)}\, t_0),$$

$$A_{(14)} = C_{(14)/0} + (C_{(3)} - 2\, C_{(7)})\, r_0 - C_{(15)}\, h_0 + C_{(16)}\, s_0 + C_{(17)}\, v_0$$

$$+2(C_{(26)}\, k_0 + C_{(29)}\, u_0 + C_{(30)}\, x_0),$$

$$A_{(15)} = C_{(15)/0} - (C_{(3)} - 2\, C_{(11)})k_0 + C_{(14)}\, h_0 + C_{(16)}\, t_0 + C_{(17)}\, w_0$$

$$-2(C_{(26)}\, r_0 - C_{(32)}\, u_0 - C_{(33)}\, x_0),$$

$$A_{(16)} = C_{(16)/0} - (C_{(3)} - 2\,C_{(20)})u_0 - C_{(14)}\,s_0 - C_{(15)}\,t_0 + C_{(17)}\,y_0$$
$$-2(C_{(29)}\,r_0 - C_{(32)}\,k_0 - C_{(35)}\,x_0),$$

$$A_{(17)} = C_{(17)/0} - (C_{(3)} - 2\,C_{(24)})x_0 - C_{(14)}\,v_0 - C_{(15)}\,w_0 - C_{(16)}\,y_0$$
$$-2(C_{(30)}\,r_0 - C_{(33)}\,k_0 - C_{(35)}\,u_0),$$

$$A_{(18)} = C_{(18)/0} + (C_{(4)} - 2\,C_{(8)})s_0 - C_{(19)}\,h_0 + C_{(20)}\,r_0 + C_{(21)}\,v_0$$
$$-2(C_{(27)}\,t_0 + C_{(29)}\,u_0 - C_{(31)}\,y_0),$$

$$A_{(19)} = C_{(19)/0} + (C_{(4)} - 2\,C_{(12)})t_0 + C_{(18)}\,h_0 - C_{(20)}\,k_0 + C_{(21)}\,w_0$$
$$-2(C_{(27)}\,s_0 + C_{(32)}\,u_0 - C_{(34)}\,y_0),$$

$$A_{(20)} = C_{(20)/0} + (C_{(4)} - 2\,C_{(16)})u_0 - C_{(18)}\,r_0 + C_{(19)}\,k_0 + C_{(21)}\,x_0$$
$$-2(C_{(29)}\,s_0 + C_{(32)}\,t_0 - C_{(35)}\,y_0),$$

$$A_{(21)} = C_{(21)/0} - (C_{(4)} - 2\,C_{(25)})y_0 - C_{(18)}\,v_0 - C_{(19)}\,w_0 - C_{(20)}\,x_0$$
$$-2(C_{(31)}\,s_0 + C_{(34)}\,t_0 + C_{(35)}\,u_0),$$

$$A_{(22)} = C_{(22)/0} + (C_{(5)} - 2\,C_{(9)})v_0 - C_{(23)}\,h_0 + C_{(24)}\,r_0 + C_{(25)}\,s_0$$
$$-2(C_{(28)}\,w_0 + C_{(30)}\,x_0 + C_{(31)}\,y_0),$$

$$A_{(23)} = C_{(23)/0} + (C_{(5)} - 2\,C_{(13)})w_0 + C_{(22)}\,h_0 - C_{(24)}\,k_0 + C_{(25)}\,t_0$$
$$-2(C_{(28)}\,v_0 + C_{(33)}\,x_0 + C_{(34)}\,y_0),$$

$$A_{(24)} = C_{(24)/0} + (C_{(5)} - 2\,C_{(17)})x_0 - C_{(22)}\,r_0 + C_{(23)}\,k_0 + C_{(25)}\,u_0$$
$$-2(C_{(30)}\,v_0 + C_{(33)}\,w_0 + C_{(35)}\,y_0),$$

$$A_{(25)} = C_{(25)/0} + (C_{(5)} - 2\,C_{(21)})y_0 - C_{(22)}\,s_0 - C_{(23)}\,t_0 - C_{(24)}\,u_0$$
$$-2(C_{(31)}\,v_0 + C_{(34)}\,w_0 + C_{(35)}\,x_0), \qquad (6.5)$$

$$A_{(26)} = C_{(26)/0} + (C_{(7)} - C_{(11)})h_0 + (C_{(10)} - C_{(14)})k_0 - (C_{(6)} - C_{(15)})r_0$$
$$+ C_{(32)}\,s_0 + C_{(29)}\,t_0 + C_{(27)}\,u_0 + C_{(33)}\,v_0 + C_{(30)}\,w_0 + C_{(28)}\,x_0,$$

$$A_{(27)} = C_{(27)/0} + (C_{(8)} - C_{(12)})h_0 - C_{(29)}\,k_0 + C_{(32)}\,r_0 - (C_{(6)} - C_{(19)})\,s_0$$

$- (C_{(10)} - C_{(18)})t_0 - C_{(26)}\, u_0 - C_{(34)}\, v_0 + C_{(31)}\, w_0 + C_{(28)}\, y_0,$

$A_{(28)} = C_{(28)/0} + (C_{(9)} - C_{(13)})h_0 - C_{(30)}\, k_0 + C_{(33)}\, r_0 + C_{(34)}\, s_0 + C_{(31)}\, t_0$

$-(C_{(6)} - C_{(23)})v_0 - (C_{(10)} - C_{(22)})w_0 - C_{(26)}\, x_0 - C_{(27)}\, y_0,$

$A_{(29)} = C_{(29)/0} - C_{(32)}\, h_0 + C_{(27)}\, k_0 - (C_{(8)} - C_{(16)})r_0 - (C_{(7)} - C_{(20)})s_0$

$-C_{(26)}\, t_0 - (C_{(14)} - C_{(18)})\, u_0 + C_{(35)}\, v_0 + C_{(31)}\, x_0 + C_{(30)}\, y_0,$

$A_{(30)} = C_{(30)/0} - C_{(33)}\, h_0 + C_{(28)}\, k_0 - (C_{(9)} - C_{(17)})r_0 + C_{(35)}\, s_0 + C_{(31)}\, u_0$

$-(C_{(7)} - C_{(24)})v_0 - C_{(26)}\, w_0 - (C_{(14)} - C_{(22)})x_0 - C_{(29)}\, y_0,$

$A_{(31)} = C_{(31)/0} - C_{(34)}\, h_0 + C_{(35)}\, r_0 - (C_{(9)} - C_{(21)})s_0 - C_{(28)}\, t_0 - C_{(30)}\, u_0$

$-(C_{(8)} - C_{(25)})\, v_0 - C_{(27)}\, w_0 - C_{(29)}\, x_0 - (C_{(18)} - C_{(22)})y_0,$

$A_{(32)} = C_{(32)/0} + C_{(29)}\, h_0 + (C_{(12)} - C_{(16)})k_0 - C_{(27)}\, r_0 - C_{(26)}\, s_0 -(C_{(11)}$

$- C_{(20)})t_0 - (C_{(15)} - C_{(19)})u_0 + C_{(35)}\, w_0 + C_{(34)}\, x_0 + C_{(33)}\, y_0,$

$A_{(33)} = C_{(33)/0} + C_{(30)}\, h_0 + (C_{(13)} - C_{(17)})\, k_0 - C_{(28)}\, r_0 + C_{(35)}\, t_0 + C_{(34)}\, u_0$

$- C_{(26)}\, v_0 -(C_{(11)} - C_{(24)})w_0 -(C_{(15)} - C_{(23)})x_0 - C_{(32)}\, y_0,$

$A_{(34)} = C_{(34)/0} + C_{(31)}\, h_0 - C_{(35)}\, k_0 - C_{(28)}\, s_0 -(C_{(13)} - C_{(21)})t_0 - C_{(33)}\, u_0$

$-C_{(27)}\, v_0 - (C_{(12)} - C_{(25)})w_0 - C_{(32)}\, x_0 - (C_{(19)} - C_{(23)})y_0,$

$A_{(35)} = C_{(35)/0} + C_{(34)}\, k_0 - C_{(31)}\, r_0 - C_{(30)}\, s_0 - C_{(33)}\, t_0 -(C_{(17)} - C_{(21)})u_0$

$-C_{(29)}\, v_0 - C_{(32)}\, w_0 - (C_{(16)} - C_{(25)})x_0 - (C_{(20)} - C_{(24)})y_0.$ (6.6).

Remark. It is important to observe that equations representing coefficients are of three different types i) Equations given for $A_{(1)}$ to $A_{(5)}$ given by (6.4), have only five terms each ii) Equations representing $A_{(6)}$ to $A_{(25)}$, given by (6.5) have nine terms each, while equations representing $A_{(26)}$ to $A_{(35)}$, given by (6.6) have 13 terms each.

Defining ${}^1P_{jk} = P_{ijk}\, m^i$, ${}^2P_{jk} = P_{ijk}\, n_{(1)}{}^l$, ${}^3P_{jk} = P_{ijk}\, n_{(2)}{}^l$, ${}^4P_{jk} = P_{ijk}\, n_{(3)}{}^l$ and ${}^5P_{jk} = P_{ijk}\, n_{(4)}{}^l$, we can obtain from equation (6.3) following equations

$$
\begin{aligned}
{}^{1}P_{jk} = L[&A_{(1)}\, m_j\, m_k + A_{(6)}\, (m_j\, n_{(1)k} + m_k\, n_{(1)j}) + A_{(7)}\, (m_j\, n_{(2)k} + m_k\, n_{(2)j}) \\
&+ A_{(8)}\, (m_j\, n_{(3)k} + m_k\, n_{(3)j}) + A_{(9)}\, (m_j\, n_{(4)k} + m_k\, n_{(4j}) + A_{(10)}\, n_{(1)j}\, n_{(1)k} \\
&+ A_{(14)}\, n_{(2)j}\, n_{(2)k} + A_{(18)}\, n_{(3)j}\, n_{(3)k} + A_{(22)}\, n_{(4)j}\, n_{(4)k} + A_{(26)}\, (n_{(1)j}\, n_{(2)k} \\
&+ n_{(1)k}\, n_{(2)j}) + A_{(27)}\, (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) + A_{(28)}\, (n_{(1)j}\, n_{(4)k} \\
&+ n_{(1)k}\, n_{(4)j}) + A_{(29)}\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) + A_{(30)}\, (n_{(2)j}\, n_{(4)k} \\
&+ n_{(2)k}\, n_{(4)j}) + A_{(31)}\, (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j})] \qquad (6.7)a
\end{aligned}
$$

$$
\begin{aligned}
{}^{2}P_{jk} = L[&A_{(2)}\, n_{(1)j}\, n_{(1)k} + A_{(6)}\, m_j\, m_k + A_{(10)}\, (m_j\, n_{(1)k} + m_k\, n_{(1)j}) \\
&+ A_{(11)}\, (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j}) + A_{(12)}\, (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) \\
&+ A_{(13)}\, (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + A_{(15)}\, n_{(2)j}\, n_{(2)k} + A_{(19)}\, n_{(3)j}\, n_{(3)k} \\
&+ A_{(23)}\, n_{(4)j}\, n_{(4)k} + A_{(26)}\, (m_j\, n_{(2)k} + m_k\, n_{(2)j}) + A_{(27)}\, (m_j\, n_{(3)k} \\
&+ m_k\, n_{(3)j}) + A_{(28)}\, (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + A_{(32)}\, (n_{(2)j}\, n_{(3)k} \\
&+ n_{(2)k}\, n_{(3)j}) + A_{(33)}\, (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) + A_{(34)}\, (n_{(3)j}\, n_{(4)k} \\
&+ n_{(3)k}\, n_{(4)j})] \qquad (6.7)b
\end{aligned}
$$

$$
\begin{aligned}
{}^{3}P_{jk} = L[&A_{(3)}\, n_{(2)j}\, n_{(2)k} + A_{(7)}\, m_j\, m_k + A_{(11)}\, n_{(1)j}\, n_{(1)k} + A_{(14)}\, (m_j\, n_{(2)k} \\
&+ m_k\, n_{(2)j}) + A_{(15)}\, (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j}) + A_{(16)}\, (n_{(2)j}\, n_{(3)k} \\
&+ n_{(2)k}\, n_{(3)j}) + A_{(17)}\, (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) + A_{(20)}\, n_{(3)j}\, n_{(3)k} \\
&+ A_{(24)}\, n_{(4)j}\, n_{(4)k} + A_{(26)}\, (m_j\, n_{(1)k} + m_k\, n_{(1)j}) + A_{(29)}\, (m_j\, n_{(3)k} \\
&+ m_k\, n_{(3)j}) + A_{(30)}\, (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + A_{(32)}\, (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) \\
&+ A_{(33)}\, (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + A_{(35)}\, (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j})] \qquad (6.7)c
\end{aligned}
$$

$$
\begin{aligned}
{}^{4}P_{jk} = L[&A_{(4)}\, n_{(3)j}\, n_{(3)k} + A_{(8)}\, m_j\, m_k + A_{(12)}\, n_{(1)j}\, n_{(1)k} + A_{(16)}\, n_{(2)j}\, n_{(2)k} \\
&+ A_{(18)}\, (m_j\, n_{(3)k} + m_k\, n_{(3)j}) + A_{(19)}\, (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) \\
&+ A_{(20)}\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) + A_{(21)}\, (n_{(3)j}\, n_{(4)k} + n_{(3)k}\, n_{(4)j}) \\
&+ A_{(25)}\, n_{(4)j}\, n_{(4)k} + A_{(27)}\, (m_j\, n_{(1)k} + m_k\, n_{(1)j}) + A_{(29)}\, (m_j\, n_{(2)k} \\
&+ m_k\, n_{(2)j}) + A_{(31)}\, (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + A_{(32)}\, (n_{(1)j}\, n_{(2)k} \\
&+ n_{(1)k}\, n_{(2)j}) + C_{(34)}\, (n_{(1)j}\, n_{(4)k} + n_{(1)k}\, n_{(4)j}) + C_{(35)}\, (n_{(2)j}\, n_{(4)k} \\
&+ n_{(2)k}\, n_{(4)j}) \qquad (6.7)d
\end{aligned}
$$

and

$$^{5}P_{jk} = L[\, A_{(5)}\, n_{(4)j}\, n_{(4)k} + A_{(9)}\, m_j\, m_k + A_{(13)}\, n_{(1)j}\, n_{(1)k} + A_{(17)}\, n_{(2)j}\, n_{(2)k}$$

$$+ A_{(21)}\, n_{(3)j}\, n_{(3)k} + A_{(22)}\, (m_j\, n_{(4)k} + m_k\, n_{(4)j}) + A_{(23)}\, (n_{(1)j}\, n_{(4)k}$$

$$+ n_{(1)k}\, n_{(4)j}) + A_{(24)}\, (n_{(2)j}\, n_{(4)k} + n_{(2)k}\, n_{(4)j}) + A_{(25)}\, (n_{(3)j}\, n_{(4)k}$$

$$+ n_{(3)k}\, n_{(4)j}) + A_{(28)}\, (m_j\, n_{(1)k} + m_k\, n_{(1)j}) + A_{(30)}\, (m_j\, n_{(2)k}$$

$$+ m_k\, n_{(2)j}) + A_{(31)}\, (m_j\, n_{(3)k} + m_k\, n_{(3)j}) + A_{(33)}\, (n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})$$

$$+ A_{(34)}\, (n_{(1)j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j}) + A_{(35)}\, (n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) \qquad (6.7)e$$

If we differentiate ($C_{ijk}\, m^i$) with respect to y^p and contract the resulting equation by y^p, we get

$$(C_{ijk}\, m^i)_{/0} = {}^{1}C_{jk/0} = P_{ijk}\, m^i + C_{ijk}\, m^i{}_{/0}.$$

Substituting the values of right- hand side by virtue of equations (4.1), (5.1) and (6.2) a, we get on simplification

$$^{1}P_{jk} = {}^{1}C_{jk/0} - {}^{2}C_{jk}\, h_0 + {}^{3}C_{jk}\, r_0 + {}^{4}C_{jk}\, s_0 + {}^{5}C_{jk}\, v_0 \qquad (6.8)\ a$$

Similarly, we get on simplification

$$^{2}P_{jk} = {}^{2}C_{jk/0} + {}^{1}C_{jk}\, h_0 - {}^{3}C_{jk}\, k_0 + {}^{4}C_{jk}\, t_0 + {}^{5}C_{jk}\, w_0, \qquad (6.8)\ b$$

$$^{3}P_{jk} = {}^{3}C_{jk/0} - {}^{1}C_{jk}\, r_0 + {}^{2}C_{jk}\, k_0 + {}^{4}C_{jk}\, u_0 + {}^{5}C_{jk}\, x_0, \qquad (6.8)\ c$$

$$^{4}P_{jk} = {}^{4}C_{jk/0} - {}^{1}C_{jk}\, s_0 - {}^{2}C_{jk}\, t_0 - {}^{3}C_{jk}\, u_0 + {}^{5}C_{jk}\, y_0, \qquad (6.8)\ d$$

$$^{5}P_{jk} = {}^{5}C_{jk/0} - {}^{1}C_{jk}\, v_0 - {}^{2}C_{jk}\, w_0 - {}^{3}C_{jk}\, x_0 - {}^{4}C_{jk}\, y_0 \qquad (6.8)\ e$$

Hence:

Theorem 6.1.: In a six-dimensional Finsler space F^6, Cartan's C-tensors are related with P-tensors given by equations (6.8) a, b, c, d, e.

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