ON THE POSITIVE PELLIAN EQUATION $y^2 = 6x^2 + 48$

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Abstract:

The binary quadratic equation represented by the Positive Pellian $y^2 = 6x^2 + 48$ is analyzed for its distinct integer solutions. Some relations among the solutions are found . Also using the solution of the above hyperbola, solutions of other choices of hyperbola and parabola are also found.

Keywords: Binary quadratic, hyperbola, parabola, pell equation, integral solutions.

2010 mathematics subject classification: 11D09

Introduction:

A Diophantine equation of degree two in two unknowns which represents a binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D positive integer and nonsqaure is studied by many mathematicians to find its integer solutions other than trivial solutions for different integer values of D.[1-2]., One may refer [3-12] for an extensive review of such problems. A hyperbola given by $y^2 = 6x^2 + 48$ is considered in this paper and infinitely many integer solutions are obtained. A few properties among the solutions are obtained.

Method of Analysis:

Consider the Positive Pell equation representing hyperbola

$$y^2 = 6x^2 + 48 \qquad (1)$$

The smallest positive integer solution (x_0, y_0) of (1) is

$$x_0 = 4 , y_0 = 12$$

To find other solutions of (1), we consider the Pell equation

$$y^2 = 6x^2 + 1 \tag{2}$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{6}} g_n$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1},$$

$$g_n = \left(5 + 2\sqrt{6}\right)^{n+1} - \left(5 - 2\sqrt{6}\right)^{n+1}, \; n = -1, 0, 1...$$

To obtain the sequence of solutions of (1), we use a lemma known as Brahmagupta lemma stated as follows:

"If (x_0, y_0) and (x_1, y_1) represent the solutions of the pell equations

$y^2 = Dx^2 + k_1$ and $y^2 = Dx^2 + k_2$ $(D > 0 \; \textit{and square free})$ respectively, then

$(x_0 y_1 + y_0 x_1,\; y_0 y_1 + D x_0 x_1)$ represents the solution of the pell equation $y^2 = Dx^2 + k_1 k_2$"

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = 2 f_n + \sqrt{6} g_n$$

$$y_{n+1} = 6 f_n + 2\sqrt{6} g_n$$

$$\Rightarrow \sqrt{6} x_{n+1} = 2\sqrt{6} f_n + 6 g_n \tag{3}$$

$$\sqrt{6} y_{n+1} = 6\sqrt{6} f_n + 12 g_n \tag{4}$$

Replacing n by $n+1$ in (3), we get

$$\sqrt{6} x_{n+2} = 2\sqrt{6} f_{n+1} + 6 g_{n+1}$$

$$\sqrt{6}x_{n+2} = 2\sqrt{6}\left(5f_n + 2\sqrt{6}g_n\right) + 6\left(5g_n + 2\sqrt{6}f_n\right)$$

$$\sqrt{6}x_{n+2} = 22\sqrt{6}f_n + 54g_n \tag{5}$$

Replacing n by $n+1$ in (5), we get

$$\sqrt{6}x_{n+3} = 22\sqrt{6}f_{n+1} + 54g_{n+1}$$

$$\sqrt{6}x_{n+3} = 22\sqrt{6}\left(5f_n + 2\sqrt{6}g_n\right) + 54\left(5g_n + 2\sqrt{6}f_n\right)$$

$$\sqrt{6}x_{n+3} = 218\sqrt{6}f_n + 534g_n \tag{6}$$

Replacing n by $n+1$ in (4), we get

$$\sqrt{6}y_{n+2} = 6\sqrt{6}f_{n+1} + 12g_{n+1}$$

$$= 6\sqrt{6}\left(5f_n + 2\sqrt{6}g_n\right) + 12\left(5g_n + 2\sqrt{6}f_n\right)$$

$$\sqrt{6}y_{n+2} = 54\sqrt{6}f_n + 132g_n \tag{7}$$

Replacing n by $n+1$ in (7), we get

$$\sqrt{6}y_{n+3} = 54\sqrt{6}f_{n+1} + 132g_{n+1}$$

$$= 54\sqrt{6}\left(5f_n + 2\sqrt{6}g_n\right) + 132\left(5g_n + 2\sqrt{6}f_n\right)$$

$$\sqrt{6}y_{n+3} = 534\sqrt{6}f_n + 1308g_n \tag{8}$$

These eqations gives non-zero distinct integer solutions of (1)

A few solutions are given in the following Table: 1

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	4	12
0	44	108

1	436	1068
2	4316	10572
3	42724	104652
4	409100	628100

The values of x_{n+1} and y_{n+1} satisfy the following The recurrence relations.

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0 \ , \ n = -1,0,1....$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0 \ , \ n = -1,0,1....$$

Some interesting relations among the solutions are given below:

1. x_{n+1} and y_{n+1} are always even.

2. Relations among the solutions

$$x_{n+2} - 2y_{n+1} - 5x_{n+1} = 0$$

$$5x_{n+2} - x_{n+1} - 2y_{n+2} = 0$$

$$49x_{n+2} - 5x_{n+1} - 2y_{n+3} = 0$$

$$x_{n+3} - 49x_{n+1} - 20y_{n+1} = 0$$

$$x_{n+3} - x_{n+1} - 4y_{n+2} = 0$$

$$49x_{n+3} - x_{n+1} - 20y_{n+3} = 0$$

$$5y_{n+1} - 12x_{n+1} - y_{n+2} = 0$$

$$49y_{n+1} + 120x_{n+1} - y_{n+3} = 0$$

$$12x_{n+1} + 49y_{n+2} - 5y_{n+3} = 0$$

$$5y_{n+3} - 12x_{n+1} - 49y_{n+2} = 0$$

$$5x_{n+3} - 49x_{n+2} - 2y_{n+1} = 0$$

$$x_{n+3} - 5x_{n+2} - 2y_{n+2} = 0$$

$$5x_{n+3} - x_{n+2} - 2y_{n+3} = 0$$

$$y_{n+1} + 12x_{n+2} - 5y_{n+2} = 0$$

$$5y_{n+1} + 120x_{n+2} - y_{n+3} = 0$$

$$5y_{n+2} - 12x_{n+2} - y_{n+3} = 0$$

$$y_{n+3} - 240x_{n+2} - y_{n+1} = 0$$

$$5y_{n+1} + 12x_{n+3} - 49y_{n+2} = 0$$

$$y_{n+1} + 120x_{n+3} - 49y_{n+3} = 0$$

$$2y_{n+2} + x_{n+3} - 5x_{n+1} = 0$$

$$y_{n+2} + 12x_{n+3} - 5y_{n+3} = 0$$

$$5y_{n+3} - 12x_{n+3} - y_{n+2} = 0$$

$$y_{n+2} - 5y_{n+1} - 12x_{n+1} = 0$$

$$y_{n+3} - y_{n+1} - 24x_{n+2} = 0$$

3. Each of the following expressions represents a Nasty Integer:

Solving (3) and (5), we get

$$f_n = \frac{1}{4}\left(x_{n+2} - 9x_{n+1}\right) \tag{9}$$

$$g_n = \frac{\sqrt{6}}{12}\left(11x_{n+1} - x_{n+2}\right) \tag{10}$$

Replacing n by $2n+1$ in (9), we get

$$f_{2n+1} = \frac{1}{4}\left(x_{2n+3} - 9x_{2n+2}\right)$$

Note that, $f_{2n+1} + 2 = f_n^2$

Now,

$$6\left[\frac{1}{4}\left(x_{2n+3} - 9x_{2n+2}\right) + 2\right] = 6 f_n^2$$

$N_1 = \frac{1}{4}\left(6x_{2n+3} - 54x_{2n+2} + 48\right)$ is a Nasty number.

The other choices of Nasty numbers are obtained by solving other pairs of equations and are given below

$$N_2 = \frac{1}{40}\left(6x_{2n+4} - 534x_{2n+2} + 480\right)$$

$$N_3 = \frac{1}{2}\left(6y_{2n+2} - 12x_{2n+2} + 24\right)$$

$$N_4 = \frac{1}{10}\left(6y_{2n+3} - 132x_{2n+2} + 120\right)$$

$$N_5 = \frac{1}{98}\left(6y_{2n+4} - 1308x_{2n+2} + 1176\right)$$

$$N_6 = \frac{1}{4}\left(54x_{2n+4} - 534x_{2n+3} + 48\right)$$

$$N_7 = \frac{1}{10}\left(54y_{2n+2} - 12x_{2n+3} + 120\right)$$

$$N_9 = \frac{1}{2}\left(54y_{2n+3} - 132x_{2n+3} + 24\right)$$

$$N_{10} = \frac{1}{10}\left(54y_{2n+4} - 1308x_{2n+3} + 120\right)$$

$$N_{11} = \frac{1}{98}\left(12x_{2n+4} - 534y_{2n+2} + 1176\right)$$

$$N_{12}=\frac{1}{10}\left(132x_{2n+4}-534y_{2n+3}+120\right)$$

$$N_{13}=\frac{1}{2}\left(1308x_{2n+4}-534y_{2n+4}+24\right)$$

$$N_{14}=\frac{1}{12}\left(66y_{2n+2}-6y_{2n+3}+144\right)$$

$$N_{15}=\frac{1}{120}\left(654y_{2n+2}-6y_{2n+4}+1440\right)$$

4. Each of the following expressions represents a Cubical Integer:

Solving (5) and (4), we get

$$f_n=\frac{1}{10}\left(9y_{n+1}-2x_{n+2}\right) \qquad (11)$$

$$g_n=\frac{\sqrt{6}}{30}\left(3x_{n+2}-11y_{n+1}\right) \qquad (12)$$

Replacing n by $3n+2$ in (11), we have

$$f_{3n+2}=\frac{1}{10}\left(9y_{3n+3}-2x_{3n+4}\right)$$

Now,

$$f_{3n+2}=f_n^3-3f_n$$

$$f_{3n+2}+3f_n=f_n^3$$

$$\Rightarrow f_n^3=\frac{1}{10}\left(9y_{3n+3}-2x_{3n+4}\right)+3\left(\frac{1}{10}\left(9y_{n+1}-2x_{n+2}\right)\right)$$

So $C_1=\frac{1}{10}\left[9y_{3n+3}-2x_{3n+4}+27y_{n+1}-6x_{n+2}\right]$ is a Cubical integer.

Solving other pairs of equations, we get different Cubical integers given below:

$$C_2=\frac{1}{4}\left[x_{3n+4}-9x_{3n+3}+3x_{n+2}-27x_{n+1}\right]$$

$C_3 = \frac{1}{40}\left[x_{3n+5} - 89x_{3n+3} + 3x_{n+3} - 267x_{n+1}\right]$

$C_4 = \frac{1}{2}\left[y_{3n+3} - 2x_{3n+3} + 3y_{n+1} - 6x_{n+1}\right]$

$C_5 = \frac{1}{10}\left[y_{3n+4} - 22x_{3n+3} + 3y_{n+2} - 66x_{n+1}\right]$

$C_6 = \frac{1}{98}\left[y_{3n+5} - 218x_{3n+3} + 3y_{n+3} - 654x_{n+1}\right]$

$C_7 = \frac{1}{4}\left[9x_{3n+5} - 89x_{3n+4} + 27x_{n+3} - 267x_{n+2}\right]$

$C_8 = \frac{1}{2}\left[9y_{3n+3} - 22x_{3n+4} + 27y_{n+1} - 66x_{n+2}\right]$

$C_9 = \frac{1}{10}\left[9y_{3n+5} - 218x_{3n+4} + 27y_{n+3} - 654x_{n+2}\right]$

$C_{10} = \frac{1}{98}\left[2x_{3n+5} - 89y_{3n+3} + 6x_{n+3} - 267y_{n+1}\right]$

$C_{11} = \frac{1}{10}\left[22x_{3n+5} - 89y_{3n+4} + 66x_{n+3} - 267y_{n+2}\right]$

$C_{12} = \frac{1}{2}\left[218x_{3n+5} - 89y_{3n+5} + 654x_{n+3} - 267y_{n+3}\right]$

$C_{13} = \frac{1}{12}\left[11y_{3n+3} - y_{3n+4} + 32y_{n+1} - 3y_{n+2}\right]$

$C_{14} = \frac{1}{120}\left[109y_{3n+3} - y_{3n+5} + 327y_{n+1} - 3y_{n+3}\right]$

$C_{15} = \frac{1}{12}\left[109y_{3n+4} - 11y_{3n+5} + 327y_{n+2} - 33y_{n+3}\right]$

5. Each of the following expressions represents a Bi-quadratic integer:

Solving (5) and (7) we get,

$$f_n = \frac{1}{2}\left(9y_{n+2} - 22x_{n+2}\right) \tag{13}$$

$$g_n = \frac{\sqrt{6}}{6}\left(27x_{n+2} - 11y_{n+2}\right) \tag{14}$$

Replacing n by $4n+3$in (13), we have

$$f_{4n+3} = \frac{1}{2}\left(9y_{4n+5} - 22x_{4n+5}\right)$$

Now,

$$f_{4n+3} + 4f_n^2 - 2 = f_n^4$$

$$\Rightarrow f_n^4 = \frac{1}{2}\left(9y_{4n+5} - 22x_{4n+5}\right) + 4\left[\frac{1}{2}\left(9y_{2n+3} - 22x_{2n+3}\right) + 2\right] - 2$$

So $B_1 = \frac{1}{2}\left[9y_{4n+5} - 22x_{4n+5} + 36y_{2n+3} - 88x_{2n+3} + 12\right]$ is a Bi-quadratic integer.

Solving other pairs of equations, we get different Bi-quadratic integers given below:

$$B_2 = \frac{1}{4}\left[x_{4n+5} - 9x_{4n+4} + 4x_{2n+3} - 36x_{2n+2} + 24\right]$$

$$B_3 = \frac{1}{40}\left[x_{4n+6} - 89x_{4n+4} + 4x_{2n+4} - 356x_{2n+2} + 80\right]$$

$$B_4 = \frac{1}{2}\left[y_{4n+4} - 2x_{4n+4} + 4y_{2n+2} - 8x_{2n+2} + 12\right]$$

$$B_5 = \frac{1}{10}\left[y_{4n+5} - 22x_{4n+4} + 4y_{2n+3} - 88x_{2n+2} + 60\right]$$

$$B_6 = \frac{1}{98}\left[y_{4n+6} - 218x_{4n+4} + 4y_{2n+4} - 872x_{2n+2} + 588\right]$$

$$B_7 = \frac{1}{4}\left[9x_{4n+6} - 89x_{4n+5} + 36x_{2n+4} - 356y_{2n+3} + 24\right]$$

$$B_8 = \frac{1}{10}\left[9y_{4n+4} - 2x_{4n+5} + 36y_{2n+2} - 8x_{2n+3} + 60\right]$$

$$B_9 = \frac{1}{10}\left[9y_{4n+6} - 218x_{4n+5} + 36y_{2n+4} - 872x_{2n+3} + 60\right]$$

$$B_{10}=\frac{1}{98}\left[89y_{4n+4}-2x_{4n+6}+356y_{2n+2}-8x_{2n+4}+588\right]$$

$$B_{11}=\frac{1}{10}\left[89y_{4n+5}-22x_{4n+6}+356y_{2n+3}-88x_{2n+4}+60\right]$$

$$B_{12}=\frac{1}{2}\left[89y_{4n+6}-218x_{4n+6}+356y_{2n+4}-872x_{2n+4}+12\right]$$

$$B_{13}=\frac{1}{12}\left[11y_{4n+4}-y_{4n+5}+44y_{2n+2}-4y_{2n+3}+72\right]$$

$$B_{14}=\frac{1}{120}\left[109y_{4n+4}-y_{4n+6}+436y_{2n+2}-4y_{2n+4}+720\right]$$

$$B_{15}=\frac{1}{12}\left[109y_{4n+5}-11y_{4n+6}+436y_{2n+3}-44y_{2n+4}+72\right]$$

6. Each of the following expressions represents a Quintic integer:

Solving (3) and (5), we get

$$f_n=\frac{1}{4}\left(x_{n+2}-9x_{n+1}\right) \tag{9}$$

$$g_n=\frac{\sqrt{6}}{12}\left(11x_{n+1}-x_{n+2}\right) \tag{10}$$

Replacing n by $5n+4$ in (9), we have

$$f_{5n+4}=\frac{1}{4}\left(x_{5n+6}-x_{5n+5}\right)$$

Now,

$$f_n^5=f_{5n+4}+5f_n^3-5f_n$$

$$\Rightarrow f_n^5=\frac{1}{4}\left(x_{5n+6}-x_{5n+5}\right)+5\left(\frac{1}{4}\left(x_{3n+4}-9x_{3n+3}+3x_{n+2}-27x_{n+2}\right)\right)$$
$$-5\left(\frac{1}{4}\left(x_{n+2}-9x_{n+1}\right)\right)$$

So $Q_1=\frac{1}{4}\left[x_{5n+6}-9x_{5n+5}+5x_{3n+4}-45x_{3n+3}+10x_{n+2}-126x_{n+1}\right]$ is a Quintic integer.

Solving other pairs of equations, we get different Quintic integers integers given below:

$$Q_2=\frac{1}{40}\left[10x_{n+3}-890x_{n+1}+5x_{3n+5}-445x_{3n+3}+x_{5n+7}-89x_{5n+5}\right]$$

$$Q_3=\frac{1}{2}\left[5y_{n+1}-20x_{n+1}+5y_{3n+3}-10x_{3n+3}+y_{5n+5}-2x_{5n+5}\right]$$

$$Q_4=\frac{1}{10}\left[10y_{n+2}-220x_{n+1}+5y_{3n+4}-110x_{3n+3}+y_{5n+6}-22x_{5n+5}\right]$$

$$Q_5=\frac{1}{98}\left[10y_{n+3}-2180x_{n+1}+5y_{3n+5}-1090x_{3n+3}+y_{5n+7}-218x_{5n+5}\right]$$

$$Q_6=\frac{1}{4}\left[90x_{n+3}-890x_{n+2}+45x_{3n+3}-445x_{3n+4}+9x_{5n+7}-89x_{5n+6}\right]$$

$$Q_7=\frac{1}{10}\left[90y_{n+1}-20x_{n+2}+45y_{3n+3}-10x_{3n+4}+9y_{5n+5}-2x_{5n+6}\right]$$

$$Q_8=\frac{1}{2}\left[90y_{n+2}-220x_{n+2}+45y_{3n+4}-110x_{3n+4}+9y_{5n+6}-22x_{5n+6}\right]$$

$$Q_9=\frac{1}{10}\left[90y_{n+3}-2180x_{n+2}+45y_{3n+5}-1090x_{3n+4}+9y_{5n+7}-218x_{5n+6}\right]$$

$$Q_{10}=\frac{1}{98}\left[20x_{n+3}-890y_{n+1}+10x_{3n+5}-445y_{3n+3}+2x_{5n+7}-89y_{5n+5}\right]$$

$$Q_{11}=\frac{1}{10}\left[220x_{n+3}-890y_{n+2}+110x_{3n+5}-445y_{3n+4}+22x_{5n+7}-89y_{5n+6}\right]$$

$$Q_{12}=\frac{1}{2}\left[890y_{n+3}-2180x_{n+3}+445y_{3n+5}-1090x_{3n+5}+89y_{5n+7}-218x_{5n+7}\right]$$

$$Q_{13}=\frac{1}{12}\left[110y_{n+1}-10y_{n+2}+55y_{3n+3}-5y_{3n+4}+11y_{5n+5}-y_{5n+6}\right]$$

$$Q_{14}=\frac{1}{120}\left[1090y_{n+1}-10y_{n+3}+545y_{3n+3}-5y_{3n+5}+109y_{5n+5}-y_{5n+7}\right]$$

$$Q_{15}=\frac{1}{12}\left[1090y_{n+2}-110y_{n+3}+545y_{3n+4}-55y_{3n+5}+109y_{5n+6}-11y_{5n+7}\right]$$

OBSERVATIONS:

1.Obtaining linear combinations among the solutions of (1), we can generate integer solutions for other choices of hyperbola which are given below:

Solving (4) and (7), we get

$$f_n=\frac{1}{12}X \tag{15}$$

$$g_n=\frac{\sqrt{6}}{24}Y \tag{16}$$

where

$$X=11y_{n+1}-y_{n+2}$$

$$Y=y_{n+2}-9y_{n+1}$$

We can note that $f_n^2-g_n^2=4$ (17)

Substituting (15) and (16) in (17), we have

$$\left[\frac{X}{12}\right]^2-\left[\frac{\sqrt{6}}{24}Y\right]^2=4$$

$$\frac{1}{144}X^2-\frac{6}{576}Y^2=4$$

$\Rightarrow 4X^2-6Y^2=2304$ which represents the Hyperbola.

The other choices of hyperbolas are presented in theTable: 2 below:

Table: 2 Hyperbolas

S.No	Hyperbolas	(X, Y)
1	$9X^2 - 6Y^2 = 576$	$(x_{n+2} - 9x_{n+1}, 11x_{n+1} - x_{n+2})$
2	$9X^2 - 6Y^2 = 57600$	$(x_{n+3} - 89x_{n+1}, 109x_{n+1} - x_{n+3})$
3	$9X^2 - 6Y^2 = 144$	$(y_{n+1} - 2x_{n+1}, y_{n+1} - 3x_{n+1})$
4	$9X^2 - 6Y^2 = 3600$	$(y_{n+2} - 22x_{n+1}, 27x_{n+1} - y_{n+2})$
5	$9X^2 - 6Y^2 = 345744$	$(y_{n+3} - 218x_{n+1}, 267x_{n+1} - y_{n+3})$
6	$9X^2 - 6Y^2 = 576$	$(9x_{n+3} - 89x_{n+2}, 109x_{n+2} - 11x_{n+3})$
7	$9X^2 - 6Y^2 = 3600$	$(9y_{n+1} - 2x_{n+2}, 3x_{n+2} - 11y_{n+1})$
8	$9X^2 - 6Y^2 = 144$	$(9y_{n+2} - 22x_{n+2}, 2x_{n+2} - 11y_{n+2})$
9	$9X^2 - 6Y^2 = 3600$	$(218x_{n+2} - 9y_{n+3}, 267x_{n+2} - 11y_{n+3})$
10	$36X^2 - 6Y^2 = 1382976$	$(2x_{n+3} - 89y_{n+1}, 6x_{n+3} - 218y_{n+1})$
11	$36X^2 - 6Y^2 = 14400$	$(22x_{n+3} - 89y_{n+2}, 54x_{n+3} - 218y_{n+2})$
12	$36X^2 - 6Y^2 = 576$	$(89y_{n+3} - 218x_{n+3}, 534x_{n+3} - 218y_{n+3})$
13	$4X^2 - 6Y^2 = 230400$	$(109y_{n+1} - y_{n+3}, y_{n+3} - 89y_{n+1})$
14	$4X^2 - 6Y^2 = 2304$	$(109y_{n+2} - 11y_{n+3}, 9y_{n+3} - 89y_{n+2})$

2. Obtaining the linear combinations among the solutions of (1), we can generate integer solutions for other choices of parabolas which are given below:

Solving (3) and (5), we get

$$f_n = \frac{1}{4} X \tag{9}$$

$$g_n = \frac{\sqrt{6}}{12} Y \tag{10}$$

where

$$X = x_{n+2} - 9x_{n+1}$$

$$Y = 11x_{n+1} - x_{n+2}$$

Replacing n by $2n+1$ in (9), we have

$$f_{2n+1} = \frac{1}{4}\left(x_{2n+3} - x_{2n+2}\right) \tag{18}$$

Note that,

$$f_{2n+1} + 2 = f_n^2$$

$$\therefore \ f_n^2 = \frac{1}{4}\left(x_{2n+3} - x_{2n+2}\right) + 2 \tag{19}$$

Since

$$f_n^2 - g_n^2 = 4 \tag{17}$$

substituting (10) and (12) in (13), we have

$$\frac{1}{4} X - \frac{6}{144} Y^2 = 4$$

$\Rightarrow 144X - 6Y^2 = 576$ which represents the Parabola.

The other choices of parabolas are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(X, Y)
1	$14400X - 6Y^2 = 57600$	$(x_{2n+4} - 89x_{2n+2} + 2,\ 109x_{n+1} - x_{n+3})$
2	$36X - 6Y^2 = 144$	$\begin{pmatrix} 2x_{2n+2} - y_{2n+2}, \\ y_{n+1} - x_{n+1} \end{pmatrix}$
3	$3894X - 13Y^2 = 60652944$	$\begin{pmatrix} 10777x_{2n+2} - y_{2n+3}, \\ y_{n+2} - 2989x_{n+1} \end{pmatrix}$
4	$5054406X - 13Y^2 = 102188080051344$	$\begin{pmatrix} 13988533x_{2n+2} - y_{2n+4}, \\ y_{n+3} - 3879721x_{n+1} \end{pmatrix}$
5	$1080X - 13Y^2 = 4665600$	$\begin{pmatrix} 3879721x_{2n+3} - 2989x_{2n+4}, \\ 829x_{n+3} - 1076041x_{n+2} \end{pmatrix}$
6	$3894X - 13Y^2 = 60652944$	$\begin{pmatrix} 13x_{2n+3} - 2989y_{2n+2}, \\ 829y_{n+1} - x_{n+2} \end{pmatrix}$
7	$6X - 13Y^2 = 144$	$\begin{pmatrix} 10777x_{2n+3} - 2989y_{2n+3}, \\ 829y_{n+2} - 2989x_{n+2} \end{pmatrix}$
8	$3894X - 13Y^2 = 60652944$	$\begin{pmatrix} 13988533x_{2n+3} - 2989y_{2n+4}, \\ 829y_{n+3} - 3879721x_{n+2} \end{pmatrix}$
9	$5054406X - 13Y^2 = 144$	$\begin{pmatrix} 13x_{2n+4} - 3879721y_{2n+2}, \\ 1076041y_{n+1} - x_{n+3} \end{pmatrix}$
10	$3894X - 13Y^2 = 60652944$	$\begin{pmatrix} 10777x_{2n+4} - 3879721y_{2n+3}, \\ 1076041y_{n+2} - 2989x_{n+3} \end{pmatrix}$

11	$6X - 13Y^2 = 144$	$\begin{pmatrix} 13988533x_{2n+4} - 3879721y_{2n+4}, \\ 1076041y_{n+3} - 3879721x_{n+3} \end{pmatrix}$
12	$14040X - 13Y^2 = 788486400$	$\begin{pmatrix} 13y_{2n+3} - 10777y_{2n+2}, \\ 2989y_{n+1} - y_{n+2} \end{pmatrix}$
13	$182239200X - 13Y^2 = 1328445040665600$	$\begin{pmatrix} 13y_{2n+4} - 13988533y_{2n+2}, \\ 3879721y_{n+1} - y_{n+3} \end{pmatrix}$
14	$14040X - 13Y^2 = 788486400$	$\begin{pmatrix} 10777y_{2n+4} - 13988533y_{2n+3}, \\ 3879721y_{n+2} - 2989y_{n+3} \end{pmatrix}$

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