Identity using strongly multiplicative functions

Uma Dixit

Assistant Professor, Department of Mathematics, University Post Graduate College, Secunderabad, Osmania University, Hyderabad-500003 (Telangana), India.

email: umadixit@gmail.com

ABSTRACT: An arithmetic function f is said to be multiplicative function in one argument if f is not identically zero and f(mn) = f(m) f(n) whenever (m,n) = 1. The objective of this paper is to present a result of multiplicative function of two variables using strongly multiplicative functions.

Keywords: Arithmetic function, Multiplicative function, strongly multiplicative functions.

---

1. INTRODUCTION

A real or complex valued function defined on the set of all positive integers is called an arithmetic function. An arithmetic function f is said to be multiplicative function in one argument if f is not identically zero and f(mn) = f(m) f(n) whenever (m,n) = 1. The function f(m,n) of two variables defined for pairs of positive integers m and n is said to be multiplicative in both the arguments m and n if f(1,1) = 1 and $f(m_1m_2,n_1n_2) = f(m_1,m_2)f(m_2,n_2)$ where $(m_1n_1,m_2n_2) = 1$.

1.1 An arithmetic function is said to be completely multiplicative function if f is not identically zero and

f(mn) = f(m)f(n) for all m, n.

1.2 Definition: *Strongly Multiplicative function*: A multiplicative arithmetic function f is said to be strongly

multiplicative function if for every prime P, we have

$$f(p) = f(p^2) = f(p^3) = \cdots \ldots \ldots \ldots \ldots$$

$f(r) = \frac{r}{\phi(r)}$ is an example of a strongly multiplicative function

1.3 Eckford cohen [3] introduced a function φ_k (n), where φ_k (n) denotes the number of non-negative integers

less than N^kwhich are relatively K-prime to N^k and $\sum_{d/n} \emptyset_k\left(N/d\right) = N^k$

1.4 The Euler Totient function φ (n) is defined to be the number of positive integers not exceeding n

which are relatively prime. The Mobius function μ (n) is defined by

$$\mu(n) = \begin{cases} 1 & if\ n = 1 \\ (-1)^k & if\ n = p_1 p_2 \dots p_k, \\ 0 & otherwise \end{cases}$$

where p_i's are distinct primes.

2. PRELIMINARIES

2.1Lemma: if $f(n)$ and $g(n)$ are multiplicative functions and

(2.1.1) $$F(m,n) = \sum_{\substack{d|n \\ (d,m)=1}} f(d)\, g\left(\frac{n}{d}\right)$$

Then $F(m,n)$ is multiplicative of two arguments m and n.

Proof: Clearly $F(1,1) = 1$. We have to prove that

$F(m_1 m_2, n_1 n_2) = F(m_1, n_1) F(m_2, n_2)$, where$m_1 n_1, m_2 n_2) = 1$

Suppose that m_1, m_2, n_1, n_2 are positive integers such that $(m_1 n_1, m_2 n_2) = 1$. Then

(2.1.2) $$F(m_1 m_2, n_1 n_2) = \sum_{\substack{d|n_1 n_2 \\ (d, m_1 m_2) = 1}} f(d)\, g\left(\frac{n_1 n_2}{d}\right)$$

Every divisor d of $n_1 n_2$ with $(d, m_1 m_2) = 1$ can be uniquely written as $d = d_1 d_2$

Where $d_1 | n_1$ and $d_2 | n_2, (d_1, m_1) = 1$ and $(d_2, m_2) = 1$

Conversely, if $d_1 | n_1,\ d_2 | n_2, (d_1, m_1) = 1\ and\ (d_2, m_2) = 1$

Then $d_1 d_2 = d$ where $d | n_1 n_2$ and $(d, m_1 m_2) = 1$

Then (2.1.2) can be written as

$$F(m_1 m_2, n_1 n_2) = \sum_{\substack{d_1|n_1 \\ d_2|n_2 \\ (d_1, m_1) = 1 \\ (d_2, m_2) = 1}} f(d_1 d_2) g\left(\frac{n_1\, n_2}{d_1\, d_2}\right)$$

$$= \sum_{d_1|n_1} f(d_1) f(d_2) g\left(\frac{n_1}{d_1}\right) g\left(\frac{n_2}{d_2}\right)$$

$d_2|n_2$
$(d_{1,}m_1) = 1$
$(d_2, m_2) = 1$

$$= \left\{\begin{matrix} \sum f(d_1) g\left(\frac{n_1}{d_1}\right) \\ d_1|n_1 \\ (d_1, m_1) = 1 \end{matrix}\right\} \left\{\begin{matrix} \sum f(d_2) g\left(\frac{n_2}{d_2}\right) \\ d_2|n_2 \\ (d_2, m_2) = 1 \end{matrix}\right\}$$

$$= F(m_1, n_1) F(m_2, n_2)$$

This proves that $F(m, n)$is multiplicative function in two arguments m and n.

2.2Lemma: Let h(n) be a multiplicative function and

(2.2.1) $G(m, n) = \mu(n)\mu\left(\frac{n}{t}\right) h\left(\frac{n}{t}\right)$where$t = (n, m)$

Then $G(m, n)$ is multiplicative function in both arguments m and n.

Proof: Clearly $G(1,1) = 1$

Suppose that m_1, m_2, n_2 and n_2 are positive integers such that $(m_1 n_1, m_2 n_2) = 1$

By (2.2.1)

$$G(m_1 m_2, n_1 n_2) = \mu(n_1 n_2)\, \mu\left(\frac{n_1 n_2}{(m_1 m_2, n_1 n_2)}\right) h\left(\frac{n_1 n_2}{(m_1 m_2, n_1 n_2)}\right)$$

$$= \mu(n_1 n_2)\, \mu\left(\frac{n_1 n_2}{(m_1, n_1)(m_2, n_2)}\right) h\left(\frac{n_1 n_2}{(m_1, n_2)(m_2, n_2)}\right)$$

$$= \mu(n_1)\mu(n_2)\, \mu\left(\frac{n_1}{m_1, n_1}\right) \mu\left(\frac{n_2}{m_2, n_2}\right) h\left(\frac{n_1}{m_1, n_1}\right) h\left(\frac{n_2}{m_2, n_2}\right)$$

$$= \left\{\mu(n_1)\, \mu\left(\frac{n_1}{(m_1, n_1)}\right) h\left(\frac{n_1}{(m_1, n_2)}\right)\right\} \left\{\mu(n_2) \mu\left(\frac{n_2}{(m_1, n_2)}\right) h\left(\frac{n_2}{(m_2, n_2)}\right)\right\}$$

$$= G(m_1, n_1) G(m_2, n_2)$$

Therefore

$G(m, n)$ is multiplicative in two arguments m and n.

3. MAIN RESULT

3.1 Theorem: If $f(n)$ is *strongly multiplicative* and $h(n)$ is *multiplicative* function such that$h(p) = f(p) - 1$

for every prime P.Then for $n \geq 1, m \geq 1$, we have

$$\sum_{\substack{d|n \\ (d,m)=1}} f(d)\mu\left(\frac{n}{d}\right) = \mu(n)\mu\left(\frac{n}{t}\right)h\left(\frac{n}{t}\right), \text{ where } t=(n,m)$$

Proof: By lemma 2.1 and lemma 2.2, both $F(m,n)$ and $G(m,n)$ are multiplicative function in m and n.

Hence it is enough to prove that $F(P^r,P^s) = G(P^r,P^s)$ for all primes P and integers $r \geq 0, s \geq 0$.

Consider

$$F(P^r,P^s) = \sum_{\substack{d|P^s \\ (d,P^r)=1}} f(d)\mu\left(\frac{P^s}{d}\right)$$

$$= \begin{cases} (f(1)\mu(P^s) + f(P)\mu(P^{s-1}) + \cdots + f(P^s)\mu(1) & if\ r=0 \\ f(1)\mu(P^s) & if\ r>0 \end{cases}$$

$$= \begin{cases} 1 & if\ r=s=0 \\ f(P)-1 & if\ s=1, r=0 \\ \mu(P^s) & if\ r>0 \end{cases}$$

$$For\ s \geq 2, r=0, \mu(1)f(P^s) + \mu(P)f(P^{s-1}) = f(P) - f(P) = 0\ , f\ is\ strongly\ multiplicative.$$

Thus,

(3.1.1) $$F(P^r,P^s) = \begin{cases} 1 & if\ r=s=0 \\ f(p)-1 & if\ s=1, r=0 \\ 0 & if\ s\geq 2, r=0 \\ \mu(P^s) & if\ r>0 \end{cases}$$

Also,

$$G(P^r,P^s) = \mu(P^s)\mu\left(\frac{P^s}{P^{\min(s,r)}}\right)h\left(\frac{P^s}{P^{\min(s,r)}}\right)$$

$$= \mu(P^s)\mu\left(P^{s-\min(s,r)}\right)h\left(P^{s-\min(s,r)}\right)$$

$$= \begin{cases} 1 & if\ r=s=0 \\ f(p)-1 & if\ s=1, r=0 \\ 0 & if\ s\geq 2, r=0 \\ \mu(P^s) & if\ r>0 \end{cases}$$

Thus

(3.1.2) $$G(P^r, P^s) = \begin{cases} 1 & if\ r = s = 0 \\ f(p) - 1 & if\ s = 1, r = 0 \\ 0 & if\ s \geq 2, r = 0 \\ \mu(P^s) & if\ r > 0 \end{cases}$$

From (3.1.1) and (3.1.2) we get $F(P^r, P^s) = G(P^r, P^s)$ and hence the result.

4. REFERENCES

[1] Apostel, T.M. An Introduction to Analytical Number Theory, Springer International Student edition, Narosa Publishing House.

[2] Eckford Cohen, Trignometric Sums in Elementary Number theory, American Mathematical Monthly, 1959, pp 105-117.

[3] Eckford Cohen, Arithmetical Inversion Formulas, Canadian J. Math., 12 (1960), pp 399-409.

[4] McCarthy, P.J. Some Remarks on Arithmetical Identities, Amerrican Math., Monthly, 67 (1956) pp 539-548.

[5] Nageswara Rao, K. On Jordan and its extension, Math Student, 29 (1961) 27.

[6] Subbarao, M.V. The Brauer – Rademacher identity, American Math, Monthly, 72 (1965), pp 135-138.