On algebraic Properties of ω- Fuzzy CI-Algebras

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Abstract- In this paper, we explored the new concept of ω –Fuzzy CI-Sub algebras and explain that new idea of Triangular Norm (ω) – Fuzzy CI-Sub algebras and prove homomorphism of Triangular Norm (ω)-Fuzzy CI-Algebras. Also we discussed the Cartesian product of Triangular Norm (ω) with Fuzzy CI-Algebras and several algebraic properties.

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I. INTRODUCTION

IN 1965, Zadeh L A [9], introduced by the concept of fuzzy sets. Meng B L [3],invented the concept of the CI - algebras in 2010. Pathak K , et al. [6], reviewed the theory of Cartesian product of BE/ CI – algebras in 2014. In 2016, Pulak Sabhapanditet al. [7] explored the new notation of Anti - Fuzzy Ideals in CI–Algebras. Sithar Selvam P M et al. [8], described the Anti Fuzzy Sub algebras and Homomorphism of CI-algebras in 2012. Meng B L [4] initiated by the concept of the Closed filters in CI - algebras in 2010. Chanwit Prabpayak [2], introduced the concept of N-Fuzzy D-Sub algebras of D-Algebras in 2018. In 2019, Anitha K and Kandaraj N [1], initiated by the concept of N-Fuzzy BH-Sub algebras of BH-Algebras. Neggers J and Kim H S [5], initiated by the concept On d-algebras in 1999. In this paper, we introduced the concept of Triangular Norm(ω)-Fuzzy CI-Sub algebras and Homomorphism of CI-Algebras established some of its results in details.

II .PRELIMINARIES

In this section we cite the fundamental definitions that will be used in the sequel.

Definition: 2.1 (Zadeh L A [9])

Let X be a non-empty set. A fuzzy subset μ of the set X is a mapping $\mu: X \rightarrow [0,1]$

Definition: 2.2 (Neggers J and Kim H S [5])

A non – empty set X with a constant 0 and a binary operation * is called a d-algebra if it satisfies the following axioms:

(i) $x * x = 0$
(ii) $0 * x = x$
(iii) $x * y = 0 \text{ and } y * x = 0 \Rightarrow x = y, \text{for all } x, y \in X$

Definition : 2.3(Meng B L [3])

A Algebraic system (X; , 1) consisting of a non –empty set X , a binary operation and a fixed element 1 , is called a CI – algebra if the following conditions are satisfied :

(i) $x * x = 1$

(ii) $1 * x = x$

(iii) $x * (y * z) = y * (x * z)$ $,\forall x, y, z \in X.$

In X we can define a binary operation ≤ be $x \leq y$ if only if $x * y = 1, \forall x, y \in X.$

Example: 2.3.1

Let $X = \{1,2,3,4\}$ be a set with a binary operation $*$ defined by the following table.

*	1	2	3	4
1	1	2	3	4
2	1	1	2	4
3	1	1	1	4
4	1	2	3	1

Then $(X,*,I)$ is a CI-algebra.

Definition 2.4

Let S be a nonempty subset of a CI-algebra X, then S is called CI-Sub algebra of X if $x * y \in S$ for all $x, y \in S.$

We define $x * y = 0$ if and only if $x \leq y,$ the $(X, \leq)$ is an ordered set. Let $\{(X,*,0/i \in I\}$ be a non empty family of CI-algebras. Then $(\Pi X_i, *, 0)$ is a CI-algebra, called the direct product of CI-algebras.

Definition: 2.5 (Meng B L [3])

Let $(X,*,I)$ be a CI-algebra. A non-empty subset I of X is called an ideal of X if it satisfies the following conditions.

(i) $x \in X \; and \; a \in I \Rightarrow x * a \in I;$

(ii) $x \in X \; and \; a, b \in I \Rightarrow (a * (b * x)) * x \in I.$

Let X be a CI-algebra. Then (i) Every ideal of X contains 1

(iii) If I is an ideal of X, then $(a * x) * x \in I, \; \forall a \in I$ and $x \in X.$

Definition: 2.6 (Pulak Sabhapandit, Biman Ch.Chetia [8])

Let X be a CI-algebra. A fuzzy set μ in X is called a fuzzy ideal of X if

(i) $\mu(x * y) \geq \mu(y), \forall x, y \in X$

(ii) $\mu\left(\left(x * (y * z)\right) * z\right) \geq min\{\mu(x), \mu(y)\}, \forall x, y, z \in X$

Definition 2.7(Sithar Selvam P M, Priya T and Ramachandran T[8])

Let μ be the fuzzy set of a set X. For a fixed $s \in [0,1],$ the set $\mu * s = \{x \in X: \mu(x) \geq s\} is$ called an upper level of μ or level subset of μ

Definition: 2.8(Sithar Selvam P M, Priya T and Ramachandran T[8])

A fuzzy set μ in a CI-algebra X is called a fuzzy sub algebra of X if $\mu(x * y) \geq min\{\mu(x), \mu(y)\}, \forall x, y \in X.$

Example: 2.8.1

1. Let $X = \{0, a, b\}$ be a set given by the following Cayley table:

*	0	a	b
0	0	0	0
a	b	0	b
b	a	a	0

Then $(X;*,0)$ is a CI-algebra, since $(b*(b*b))*b = (b*0)*b = a*b = b \neq 0$. We define a fuzzy *set* $\mu : X \to [0,1]$ *by* $\mu(0) = 0.8, \mu(x) = 0.03$, where for all $\neq 0$. It is easy to see that μ is a fuzzy CI-sub algebra of X.

2. Let $X = \{0, a, b\}$ be a set

Define $x * y = \begin{cases} 0 \; if \; x \leq y \\ x - y \; if \; y < x \end{cases}$

Then $(X;*,0)$ is an infinite CI-algebra. Define a fuzzy set $\mu : X \to [0,1]$ *by* $\mu(0) = s_1, \mu(x) = s_2$, where for all $x \neq 0$ and $s_1 > s_2$. Then μ is a fuzzy CI-sub algebra of X.

Definition: 2.9

A triangular norm is a binary operation N on the unit interval $[0,1]$ which is Commutative, Associative, Monotone and has 1 as neutral element. It is a function $N: [0,1]^2 \to [0,1]$ satisfying the following properties.

(i) $N(x,1) = x$

(ii) $N(x,y) = N(y,x)$

(iii) $N(x, N(y,z)) = N(N(x,y), z)$

(iv) $N(x,y) \leq N(x,z)$ whenever $y \leq Z, \forall\, x,y,z \in [0,1]$.

Definition: 2.10

Let μ and δ be the fuzzy sets in X. The Cartesian product $\mu \times \delta : X \times X \to [0,1]$ is defined by $(\mu \times \delta)(x,y) = min\{\mu(x), \delta(y)\}, \forall\, x,y \in X$.

III. TRIANGULAR NORM (ω)-FUZZY CI-SUB ALGEBRAS IN CI-ALGEBRAS

Definition: 3.1

A fuzzy set ξ in a CI-Algebra of κ. Then ξ is called a Fuzzy CI-Sub algebra of κ with respect to a Triangular Norm(ω) ω-Fuzzy CI-Sub algebra of κ if the following conditions are satisfied:

$$\xi(\pi * \vartheta) \geq \{\xi(\pi)\ \omega\ \xi(\vartheta)\}, \forall\, \pi, \vartheta \in \kappa.$$

Example: 3.1.1

Let $\kappa = \{0, a, b, c\}$be a set given by the following Cayley table:

Let $*$ with a binary operation defined by

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Then κ is CI-Algebras. Define a fuzzy set ξ of κ by $\xi(0) = 0.6$ and $\xi(\pi) = 0.3\ \forall\, \pi \neq 0$.

We define a Triangular Norm,

$$\omega: [0,1]^2 \to [0,1] \text{ by } \omega(\pi, \vartheta) = max\{\pi + \vartheta - 1, 0\}, \forall\, \pi, \vartheta \in \kappa.$$

Theorem: 3.2

If κ be a CI-Algebra. Let ξ be an Triangular Norm (ω)-Fuzzy CI-Sub algebra of κ and $\acute{\eta} \in [0,1]$. Then we have to following conditions are;

(i) Let $\acute{\eta} = 1$, then the upper level of ξ in κ, $\Delta(\xi, \acute{\eta})$ is either empty or a CI-Sub algebra of κ.

(ii) Let $\omega = \wedge$, then $\Delta(\xi, \acute{\eta})$ is either empty or a Fuzzy CI-Sub algebra of κ.

(iii) Let $\omega = \wedge$, then $\xi(0) \geq \xi(\pi)$, $\forall\, \pi \in \kappa$.

Proof:

(i) Assume that $\Delta(\xi, 1)$ is not empty.

Let $\pi, \vartheta \in \Delta(\xi, 1)$. Thus we have $\xi(\pi) \geq 1$ and $\xi(\vartheta) \geq 1$.

Since ξ is an ω -Fuzzy CI-Sub algebra of κ.

Now $\xi(\pi * \vartheta) \geq \{\xi(\pi)\ \omega\ \xi(\vartheta)\}$

$\geq \{1\ \omega\ 1\}$

$= 1.$

$\Rightarrow \pi * \vartheta \in \Delta(\xi, 1).$

$\therefore\ \Delta(\xi, 1)$ is a Fuzzy CI-Sub algebra.

(ii) Let as assume that $\Delta(\xi, \acute{\eta})$ is not empty

Let $\pi, \vartheta \in \Delta(\xi, \acute{\eta})$. Then we have $\xi(\pi) \geq \acute{\eta}$ and $\xi(\vartheta) \geq \acute{\eta}$.

This follows that,

$$\xi(\pi * \vartheta) \geq \{\xi(\pi)\ \omega\ \xi(\vartheta)\}$$

$\geq \{\xi(\pi) \wedge \xi(\vartheta)\}$

$\geq \{\pi \wedge \pi\} = \pi$

$\Rightarrow \pi * \vartheta \in \Delta(\xi, \acute{\eta}),$

$\therefore\ \Delta(\xi, \acute{\eta})$ is a CI-Sub algebra.

(iii) Let $\pi * \pi = 0$, we have

$\xi(0) = \xi(\pi * \pi)$

$\geq \{\xi(\pi)\ \ \omega\ \xi(\pi)\}$

$= \{\xi(\pi) \wedge \xi(\pi)\}$

$= \xi(\pi).$

$\Rightarrow \xi(0) \geq \xi(\pi), \forall\, \pi \in \kappa.$ ■

Theorem: 3.3

If ξ be a fuzzy set of CI-Algebra κ is an ω -Fuzzy CI-Sub algebra if and only if for every $\acute{\eta} \in [0,1]$, $\xi^{\acute{\eta}}$ is either empty or a Sub algebra of κ.

Proof:

Let ξ be an ω -Fuzzy CI-Sub algebra of κ and $\xi^{\acute{\eta}} \neq \emptyset$.

Then for all , $\vartheta \in \xi^{\acute{\eta}}$,

We have,

$$\xi(\pi * \vartheta) \geq \{\xi(\pi)\ \omega\ \xi(\vartheta)\} \geq \acute{\eta}.$$

$$\therefore \pi * \vartheta \in \xi^{\acute{\eta}} .$$

Hence $\xi^{\acute{\eta}}$ is a ω -Fuzzy CI-Sub algebra of κ.

Converse

Assume that $\pi * \vartheta \in \kappa$.

Take $\acute{\eta} = \{\xi(\pi)\ \omega\ \xi(\vartheta)\}$.

Then by assumption $\xi^{\acute{\eta}}$ is a ω - fuzzy CI-Sub algebra of κ.

$$\Rightarrow\ \pi * \vartheta \in \xi^{\acute{\eta}}.$$

$$\therefore \xi(\pi * \vartheta) \geq \acute{\eta} = \{\xi(\pi)\ \omega\ \xi(\vartheta)\}.$$

Hence ξ is an ω -Fuzzy CI-Sub algebra of κ. ∎

IV.HOMOMORPHISM OF TRIANGULAR NORM(ω)-FUZZY CI-SUB ALGEBRAS AND IDEALS IN CI-ALGEBRA

In this section, we discussed about the new concept of homomorphism of Triangular Norm(ω)-Fuzzy CI-Sub algebra and Ideal in CI-Algebra. And results are stated and proved.

Definition: 4.1

Let $\psi: \Omega \to \Upsilon$ be an endomorphism and ξ be a ω -Fuzzy set in κ. We define a ω -Fuzzy set in κ by ξ_ψ in δ as $\xi_\psi(\pi) = \xi(\psi(\pi)), \forall\ \pi \in \kappa.$

Definition: 4.2

Let ψ be a mapping on a set κ, and ξ be a ω -Fuzzy set of $\psi(\kappa)$. Then the ω -Fuzzy set ξ_ψ is called preimage of ξ under ψ.

Definition: 4.3

Let κ be a CI-algebra. A fuzzy set ξ in κ is called a ω - Fuzzy CI- ideal of κ if

(i) $\xi(\pi * \vartheta) \geq \xi(\vartheta), \forall\ \pi, \vartheta \in \kappa$

(ii) $\xi\left(\left(\pi * (\vartheta * \tau)\right) * \tau\right) \geq \{\xi(\pi)\ \omega\ \xi(\vartheta)\}, \forall\ \pi, \vartheta, \tau \in \kappa.$

Definition: 4.4

An $(\Omega, *, \nu)$ and $(\Upsilon, \circ, \nu')$ be CI-algebras. A mapping $\psi: \Omega \to \Upsilon$ is said to be a homomorphism if $\psi(\pi * \vartheta) = \psi(\pi) \circ \psi(\vartheta), \forall\ \pi, \vartheta \in \kappa.$

Lemma: 4.4

The epimorphism preimage of a ω -Fuzzy CI-Sub algebra of κ is a ω -Fuzzy CI-Sub algebra.

Theorem: 4.5

If $\psi: \Omega \to \Upsilon$ be an epimorphism of CI-algebras and ξ be an ω -Fuzzy CI-Sub algebra of Y. Then ξ_ψ is an ω -Fuzzy CI-Sub algebra of κ.

Proof:

Let $\psi: \Omega \to \Upsilon$ be an epimorphism of CI-algebra and ξ_ψ is obviously a ω -Fuzzy set of κ.

Let $\pi, \vartheta \in \kappa$. Then $\xi_\psi(\pi * \vartheta) = \xi(\psi(\pi * \vartheta))$

$$= \xi((\psi(\pi) * \psi(\vartheta))$$
$$\geq \{\xi(\psi(x))\ \omega\ \xi(\psi(\vartheta))\}$$
$$\geq \{\xi_\psi(\pi)\ \omega\ \xi_\psi(\vartheta)\}$$

Hence ξ_ψ is an ω -Fuzzy CI-Sub algebra of κ. ∎

Theorem: 4.6

If ψ be an endomorphism of a CI-Algebra of κ. Then ξ is an ω -Fuzzy CI-Ideal of κ, then so is ξ_ψ.

Proof:

Let ψ be an endomorphism of a CI-Algebra of κ.

Let $\pi, \vartheta, \tau \in \kappa$

$$\xi_\psi\{(\pi * (\vartheta * \tau)) * \tau\} = \xi\{\psi((\pi * (\vartheta * \tau)) * \tau\}$$
$$= \xi\{\psi(\pi * (\vartheta * \tau)) * \psi(\tau)\}$$
$$= \xi\{(\psi(\pi) * \psi(\vartheta * \tau)) * \psi(\tau)\}$$
$$= \xi\left\{\left(\psi(\pi) * (\psi(\vartheta) * \psi(\tau))\right) * \psi(\tau)\right\}$$
$$\geq \{\xi(\psi(\pi))\ \omega\ \xi(\psi(\vartheta))\}$$
$$= \{\xi_\psi(\pi)\ \omega\ \xi_\psi(\vartheta)\}$$

$$\therefore \xi_\psi\{(\pi * (\vartheta * \tau)) * \tau\} \geq \{\xi_\psi(\pi)\ \omega\ \xi_\psi(\vartheta)\}$$

Hence ξ_ψ is an ω -Fuzzy CI-Ideal of κ. ■

Theorem: 4.7

Let $\psi: \Omega \to \Upsilon$ be an homomorphism of CI-Algebra. If ξ is a ω -Fuzzy CI-Ideal of Υ then ξ_ψ is an ω -Fuzzy CI-Ideal of κ.

Proof:

Let $\pi, \vartheta, \tau \in \kappa$.

Now, $\xi_\psi\{(\pi * (\vartheta * \tau)) * \tau\} = \xi\{\psi((\pi * (\vartheta * \tau)) * \tau\}$

$$= \xi\{\psi(\pi * (\vartheta * \tau)) \circ \psi(\tau)\}$$
$$= \xi\{(\psi(\pi) \circ \psi(\vartheta * \tau)) \circ \psi(\tau)\}$$
$$= \xi\{(\psi(\pi)\circ(\psi(\vartheta)\circ\psi(\tau)))\circ\psi(\tau)\}$$
$$\geq \{\xi(\psi(\pi))\ \omega\ \xi(\psi(\vartheta))\}$$
$$= \{\xi_\psi(\pi)\ \omega\ \xi_\psi(\vartheta)\}$$
$$\therefore \xi_\psi\{(\pi * (\vartheta * \tau)) * \tau\} \geq \{\xi_\psi(\pi)\ \omega\ \xi_\psi(\vartheta)\}.$$

Hence ξ_ψ is an ω -Fuzzy CI-Ideal of κ. ■

V. TRIANGULAR NORM(ω)-FUZZY CI-SUB ALGEBRAS IN CARTESIAN PRODUCT OF CI- ALGEBRAS

In this section , we introduced the new notion of Triangular Norm(ω)-Fuzzy CI-Sub algebras in Cartesian product of CI-Algebras.

Definition: 5.1

An ξ_1 and ξ_2 be ω -Fuzzy CI-Sub algebra of a CI-Algebra κ. The direct product of ω -Fuzzy CI-Sub algebras ξ_1 and ξ_2 defined by $\xi_1 \times \xi_2(\pi, \vartheta) = \{\xi_1(\pi)\ \omega\ \xi_2(\vartheta)\}, \forall\ \pi, \vartheta \in \kappa$.

Theorem: 5.2

If ξ_1 and ξ_2 be ω -Fuzzy CI-Sub algebra of a CI-Algebra κ. Let κ be a CI-Algebra. Then $\xi_1 \times \xi_2$ is a ω -Fuzzy CI-Sub algebra of κ.

Proof:

Let ξ_1 and ξ_2 be ω -Fuzzy CI-Sub algebra of a CI-Algebra κ.

Put $\xi = \xi_1 \times \xi_2$, and let $\pi, \vartheta \in \kappa$.

Then, we have

$$\xi((\pi_1, \pi_2) * (\vartheta_1, \vartheta_2)) = \xi(\pi_1 * \vartheta_1, \pi_2 * \vartheta_2)$$
$$= (\xi_1 \times \xi_2)(\pi_1 * \vartheta_1, \pi_2 * \vartheta_2)$$
$$= \{\xi_1(\pi_1 * \vartheta_1)\ \omega\ \xi_2(\pi_2 * \vartheta_2)\}$$
$$\geq \{(\xi_1(\pi_1)\ \omega\ \xi_1(\vartheta_1))\ \omega\ (\xi_2(\pi_2) \omega\ \xi_2(\vartheta_2))\}$$
$$= \{(\xi_1(\pi_1)\ \omega\ \xi_2(\pi_2))\ \omega\ (\xi_1(\vartheta_1)\ \omega\ \xi_2(\vartheta_2))\}$$
$$= \{((\xi_1 \times \xi_2)(\pi_1, \pi_2))\ \omega\ ((\xi_1 \times \xi_2)(\vartheta_1, \vartheta_2))\}$$
$$= \{\xi(\pi_1, \pi_2)\ \omega\ \xi(\vartheta_1, \vartheta_2)\}.$$

Hence $\xi_1 \times \xi_2$ is an ω -Fuzzy CI-Sub algebra of κ. ■

VI. CONCLUSION

Hence we have discussed the Triangular Norm(ω)-Fuzzy CI-Sub algebras in CI-Algebra. Which adds another dimension to the defined Homomorphism of Triangular Norm(ω)-Fuzzy CI-Sub algebras and Ideals in CI-Algebra and Cartesian Product of Triangular Norm(ω)-Fuzzy CI-Sub algebras in CI– Algebra. This concept can further be generalized to fuzzy translation and multiplication of Triangular Norm(ω)-Fuzzy CI-Sub algebra of CI-Algebra , Doubt Triangular Norm(ω)- Q-Fuzzy CI-Sub algebras and homomorphism of CI-algebras in our future work.

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