ON THE NEGATIVE PELLIAN EQUATION $y^2 = 13x^2 - 12$

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ABSTRACT

The hyperbola $y^2 = 13x^2 - 12$ is studied for its different solutions in integers. Some remarkable relations among the solutions are given. Also, integer solutions for other choices of hyperbolas and parabolas based on a given solution of the hyperbola under consideration are exhibited.

Keywords: Second degree with two unknowns, hyperbola, parabola, pell equation, solutions in integers.

1. INTRODUCTION

Many mathematicians analysed the binary quadratic diophantine equation of the form $y^2 = Dx^2 - N \ (N > 0)$, where D is a non-square positive integer [1-3]. The above equation is called the Negative form of the pell equation or related pell equation. It is worth to remind that the above equation is solvable only for certain values of D. In particular, one may refer [4-12].

This paper concerns with the equation $y^2 = 13x^2 - 12$ for determining different sets of solutions in integers and exhibits some remarkable relations between the solutions.

2. METHOD OF ANALYSIS

The binary quadratic equation to be solved is

$$y^2 = 13x^2 - 12 \tag{1}$$

whose initial solution is $x_0 = 1$, $y_0 = 1$

Now consider the fundamental positive pell equation

$$y^2 = 13x^2 + 1 \tag{2}$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{13}} g_n$$
$$\tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(649 + 180\sqrt{13}\right)^{n+1} + \left(649 - 180\sqrt{13}\right)^{n+1},$$

$$g_n = \left(649 + 180\sqrt{13}\right)^{n+1} - \left(649 - 180\sqrt{13}\right)^{n+1}, \ n = -1, 0, 1...$$

Applying the lemma of Brahmagupta between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{1}{2\sqrt{13}} g_n$$

$$y_{n+1} = \frac{1}{2} f_n + \frac{\sqrt{13}}{2} g_n$$

$$\Rightarrow \ 2\sqrt{13}x_{n+1} = \sqrt{13} f_n + g_n \tag{3}$$

$$2\sqrt{13}y_{n+1} = \sqrt{13} f_n + 13g_n \tag{4}$$

Replacing n by $n+1$ in (3), we get

$$2\sqrt{13}x_{n+2} = \sqrt{13} f_{n+1} + g_{n+1}$$

$$2\sqrt{13}x_{n+2} = \sqrt{13}\left(649 f_n + 180\sqrt{13}g_n\right) + \left(649g_n + 180\sqrt{13} f_n\right)$$

$$2\sqrt{13}x_{n+2} = 829\sqrt{13} f_n + 2989g_n \tag{5}$$

Replacing n by $n+1$ in (5), we get

$$2\sqrt{13}x_{n+3} = 829\sqrt{13} f_{n+1} + 2989g_{n+1}$$

$$2\sqrt{13}x_{n+3} = 829\sqrt{13}\left(649 f_n + 180\sqrt{13}g_n\right) + 2989\left(649g_n + 180\sqrt{13} f_n\right)$$

$$2\sqrt{13}x_{n+3} = 1076041\sqrt{13} f_n + 3879721g_n \tag{6}$$

Replacing n by $n+1$ in (4), we get

$$2\sqrt{13}y_{n+2} = \sqrt{13} f_{n+1} + 13g_{n+1}$$

$$= \sqrt{13}\left(649 f_n + 180\sqrt{13}g_n\right) + 13\left(649g_n + 180\sqrt{13} f_n\right)$$

$$2y_{n+2} = 2989\sqrt{13}f_n + 10777g_n \qquad (7)$$

Replacing n by $n+1$ in (7), we get

$$2y_{n+3} = 2989\sqrt{13}f_{n+1} + 10777g_{n+1}$$

$$= 2989\sqrt{13}\left(649f_n + 180\sqrt{13}g_n\right) + 10777\left(649g_n + 180\sqrt{13}f_n\right)$$

$$2y_{n+3} = 3879721\sqrt{13}f_n + 13988533g_n \qquad (8)$$

The recurrence relations satisfied by the values of x_{n+1} and y_{n+1} are respectively,

$$x_{n+3} - 1298x_{n+2} + x_{n+1} = 0 \ , \ n = -1,0,1....$$

$$y_{n+3} - 1298y_{n+2} + y_{n+1} = 0 \ , \ n = -1,0,1....$$

A few numerical examples are given in the following Table: 1

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	1
0	829	2989
1	1076041	3879721
2	1396700389	5035874869

Some remarkable relations between the solutions are given below:

1. x_{n+1} and y_{n+1} are always odd.

2. Relations among the solutions

- $x_{n+2} - 180y_{n+1} - 649x_{n+1} = 0$
- $649x_{n+2} - x_{n+1} - 180y_{n+2} = 0$
- $842401x_{n+2} - 649x_{n+1} - 180y_{n+3} = 0$
- $x_{n+3} - 842401x_{n+1} - 233640y_{n+1} = 0$
- $x_{n+3} - x_{n+1} - 360y_{n+2} = 0$
- $2340x_{n+1} + 649y_{n+1} - y_{n+2} = 0$
- $3037320x_{n+1} + 842401y_{n+1} - y_{n+3} = 0$
- $2340x_{n+1} + 842401y_{n+2} - 649y_{n+3} = 0$

- $649x_{n+3} - 842401x_{n+2} - 180y_{n+1} = 0$
- $x_{n+3} - 649x_{n+2} - 180y_{n+2} = 0$
- $649x_{n+3} - 180y_{n+3} - x_{n+2} = 0$
- $2340x_{n+2} + y_{n+1} - 649y_{n+2} = 0$
- $4680x_{n+2} + y_{n+1} - y_{n+3} = 0$
- $2340x_{n+2} + 649y_{n+2} - y_{n+3} = 0$
- $649x_{n+3} - 180y_{n+1} - 842401x_{n+2} = 0$
- $649y_{n+1} + 2340x_{n+3} - 842401y_{n+2} = 0$
- $y_{n+1} + 3037320x_{n+3} - 842401y_{n+3} = 0$
- $y_{n+2} + 2340x_{n+3} - 649y_{n+3} = 0$
- $842401y_{n+2} - 649x_{n+1} - 2340x_{n+3} = 0$
- $649x_{n+3} - 180y_{n+3} - x_{n+2} = 0$

3. Nasty Number:

Solving (3) and (5), we get

$$f_n = \frac{1}{1080}\left(2989x_{n+1} - x_{n+2}\right) \tag{9}$$

$$g_n = \frac{\sqrt{13}}{1080}\left(x_{n+2} - 829x_{n+1}\right) \tag{10}$$

Replacing n by $2n+1$ in (9), we have

$$f_{2n+1} = \frac{1}{1080}\left(2989x_{2n+2} - x_{2n+3}\right)$$

Note that, $f_{2n+1} + 2 = f_n^2$

Now,

$$6\left[\frac{1}{1080}\left(2989x_{2n+2} - x_{2n+2}\right) + 2\right] = 6f_n^2$$

$$\therefore \frac{1}{1080}\left(17934x_{2n+2} - 6x_{2n+3} + 12960\right) \text{ is a Nasty number.}$$

For elegance and clarity, the other choices of Nasty numbers are presented below:

- $\frac{1}{1401840}\left(23278326x_{2n+2}-6x_{2n+4}+16822080\right)$
- $\frac{1}{6}\left(78x_{2n+2}-6y_{2n+2}+72\right)$
- $\frac{1}{3894}\left(64662x_{2n+2}-6y_{2n+3}+46728\right)$
- $\frac{1}{5054406}\left(83931198x_{2n+2}-6y_{2n+4}+60652872\right)$
- $\frac{1}{1080}\left(23278326x_{2n+3}-17934x_{2n+4}+12960\right)$
- $\frac{1}{3894}\left(78x_{2n+3}-17934y_{2n+2}+46728\right)$
- $\frac{1}{6}\left(64662x_{2n+3}-17934y_{2n+3}+72\right)$
- $\frac{1}{3894}\left(83931198x_{2n+3}-17934y_{2n+4}+46728\right)$
- $\frac{1}{5054406}\left(78x_{2n+4}-23278326y_{2n+2}+60652872\right)$
- $\frac{1}{3894}\left(10777x_{2n+4}-3879721y_{2n+3}+46728\right)$
- $\frac{1}{6}\left(83931198x_{2n+4}-232783267y_{2n+4}+72\right)$
- $\frac{1}{14040}\left(78y_{2n+3}-64662y_{2n+2}+168480\right)$
- $\frac{1}{18223920}\left(78y_{2n+4}-83931198y_{2n+2}+218687040\right)$
- $\frac{1}{14040}\left(64662x_{2n+4}-83931198y_{2n+3}+168480\right)$

4. Cubical Integer:

Solving (3) and (5), we get

$$f_n=\frac{1}{1080}\left(2989x_{n+1}-x_{n+2}\right) \tag{9}$$

$$g_n=\frac{\sqrt{13}}{1080}\left(x_{n+2}-829x_{n+1}\right) \tag{10}$$

Replacing n by $3n+2$ in (9), we have

$$f_{3n+2} = \frac{1}{1080}\left(2989x_{3n+3} - x_{3n+4}\right)$$

Now,

$$f_{3n+2} = f_n^3 - 3f_n$$

$$f_{3n+2} + 3f_n = f_n^3$$

$$\Rightarrow f_n^3 = \frac{1}{1080}\left(2989x_{3n+3} - x_{3n+4}\right) + 3\left(\frac{1}{1080}\left(2989x_{n+1} - x_{n+2}\right)\right)$$

$\therefore \frac{1}{1080}\left[2989x_{3n+3} - x_{3n+4} + 8967x_{n+1} - 3x_{n+2}\right]$ is a Cubical integer.

For elegance and clarity, the other choices of Cubical integers are presented below:

- $\frac{1}{1401840}\left[3879721x_{3n+3} - x_{3n+5} + 11639163x_{n+1} - 3x_{n+3}\right]$
- $\frac{1}{6}\left[13x_{3n+3} - y_{3n+3} + 39x_{n+1} - 3y_{n+1}\right]$
- $\frac{1}{3894}\left[10777x_{3n+3} - y_{3n+4} + 32331x_{n+1} - 3y_{n+2}\right]$
- $\frac{1}{5054406}\left[13988533x_{3n+3} - y_{3n+5} + 41965599x_{n+1} - 3y_{n+3}\right]$
- $\frac{1}{1080}\left[3879721x_{3n+4} - 2989x_{3n+5} + 11639163x_{n+2} - 8967x_{n+3}\right]$
- $\frac{1}{3894}\left[13x_{3n+4} - 2989y_{3n+3} + 39x_{n+2} - 8967y_{n+1}\right]$
- $\frac{1}{6}\left[10777x_{3n+4} - 2989y_{3n+4} + 32331x_{n+2} - 8967y_{n+2}\right]$
- $\frac{1}{3894}\left[13988533x_{3n+4} - 2989y_{3n+5} + 41965599x_{n+2} - 8967y_{n+3}\right]$
- $\frac{1}{5054406}\left[13x_{3n+5} - 3879721y_{3n+3} + 39x_{n+3} - 11639163y_{n+1}\right]$
- $\frac{1}{3894}\left[10777x_{3n+5} - 3879721y_{3n+4} + 32331x_{n+3} - 11639163y_{n+2}\right]$
- $\frac{1}{6}\left[13988533x_{3n+5} - 3879721y_{3n+5} + 41965599x_{n+3} - 11639163y_{n+3}\right]$
- $\frac{1}{14040}\left[10777y_{3n+3} - 13y_{3n+4} + 32331y_{n+1} - 39y_{n+2}\right]$

- $\frac{1}{18223920}\left[13y_{3n+5}-13988533y_{3n+3}+39y_{n+3}-41965599y_{n+1}\right]$
- $\frac{1}{14040}\left[10777y_{3n+5}-13988533y_{3n+4}+32331y_{n+3}-41965599y_{n+2}\right]$

5. Bi-quadratic integer:

Solving (3) and (5) we get,

$$f_n=\frac{1}{1080}\left(2989x_{n+1}-x_{n+2}\right) \quad (9)$$

$$g_n=\frac{\sqrt{13}}{1080}\left(x_{n+2}-829x_{n+1}\right) \quad (10)$$

Replacing n by $4n+3$ in (9), we have

$$f_{4n+3}=\frac{1}{1080}\left(2989x_{4n+4}-x_{4n+5}\right)$$

Now,

$$f_{4n+3}+4f_n^2-2=f_n^4$$

$$\Rightarrow f_n^4=\frac{1}{1080}\left(2989x_{4n+4}-x_{4n+5}\right)+4\left[\frac{1}{1080}\left(2989x_{n+1}-x_{n+2}\right)+2\right]-2$$

$\therefore \frac{1}{1080}\left[2989x_{4n+4}-x_{4n+5}+11956x_{2n+2}-4x_{2n+3}+6480\right]$ is a Bi-quadratic integer.

For elegance and clarity, the other choices of Bi-quadratic integers are presented below:

- $\frac{1}{1401840}\left[3879721x_{4n+4}-x_{4n+6}+15518884x_{2n+2}-4x_{2n+4}+8411040\right]$
- $\frac{1}{6}\left[13x_{4n+4}-y_{4n+4}+52x_{2n+2}-4y_{2n+2}+36\right]$
- $\frac{1}{3894}\left[10777x_{4n+4}-y_{4n+5}+43108x_{2n+2}-4y_{2n+3}+23364\right]$
- $\frac{1}{5054406}\left[13988533x_{4n+4}-y_{4n+6}+55954132x_{2n+2}-4y_{2n+4}+30326436\right]$
- $\frac{1}{1080}\left[3879721x_{4n+5}-2989x_{4n+6}+15518884x_{2n+3}-11956x_{2n+4}+6480\right]$
- $\frac{1}{3894}\left[13x_{4n+5}-2989y_{4n+4}+52x_{2n+3}-11956y_{2n+2}+23364\right]$
- $\frac{1}{6}\left[10777x_{4n+5}-2989y_{4n+5}+43108x_{2n+3}-11956y_{2n+3}+36\right]$
- $\frac{1}{3894}\left[13988533x_{4n+5}-2989y_{4n+6}+55954132x_{2n+3}-11956y_{2n+4}+23364\right]$

- $\frac{1}{5054406}\left[13x_{4n+6}-3879721y_{4n+4}+52x_{2n+4}-15518884y_{2n+2}+30326436\right]$
- $\frac{1}{3894}\left[10777x_{4n+6}-3879721y_{4n+5}+43108x_{2n+4}-15518884y_{2n+3}+23364\right]$
- $\frac{1}{6}\left[13988533x_{4n+6}-3879721y_{4n+6}+55954132x_{2n+4}-15518884y_{2n+4}+36\right]$
- $\frac{1}{14040}\left[10777y_{4n+4}-13y_{4n+5}+52y_{2n+3}-43108y_{2n+2}+84240\right]$
- $\frac{1}{18223920}\left[13y_{4n+6}-13988533y_{4n+4}+48y_{2n+4}-55954132y_{2n+2}+109343520\right]$
- $\frac{1}{14040}\left[10777y_{4n+6}-13988533y_{4n+5}+43108y_{2n+4}-55954132y_{2n+3}+84240\right]$

6. Quintic integer:

Solving (3) and (5), we get

$$f_n=\frac{1}{1080}\left(2989x_{n+1}-x_{n+2}\right) \tag{9}$$

$$g_n=\frac{\sqrt{13}}{1080}\left(x_{n+2}-829x_{n+1}\right) \tag{10}$$

Replacing n by $5n+4$ in (9), we have

$$f_{5n+4}=\frac{1}{1080}\left(2989x_{5n+5}-x_{5n+6}\right)$$

Now,

$$f_n^5=f_{5n+4}+5f_n^3-5f_n$$

$$\Rightarrow f_n^5=\frac{1}{1080}\left(2989x_{5n+5}-x_{5n+6}\right)+5\left(\frac{1}{1080}\left(2989x_{3n+3}-x_{3n+4}+8967x_{n+1}-3x_{n+2}\right)\right)$$
$$-5\left(\frac{1}{1080}\left(2989x_{n+1}-x_{n+2}\right)\right)$$

$\therefore \frac{1}{1080}\left[2989x_{5n+5}-x_{5n+6}+14945x_{3n+3}-5x_{3n+4}+29890x_{n+1}-10x_{n+2}\right]$ is a Quintic integer.

For elegance and clarity, the other choices of Quintic integers are presented below:

- $\frac{1}{1401840}\left[\begin{array}{l}38797210x_{n+1}-10x_{n+3}+19398605x_{3n+3}-5x_{3n+5}\\+3879721x_{5n+5}-x_{5n+7}\end{array}\right]$
- $\frac{1}{6}\left[130x_{n+1}-10y_{n+1}+65x_{3n+3}-5y_{3n+3}+13x_{5n+5}-y_{5n+5}\right]$
- $\frac{1}{3894}\left[107770x_{n+1}-10y_{n+2}+53885x_{3n+3}-5y_{3n+4}+10777x_{5n+5}-y_{5n+6}\right]$

- $\frac{1}{5054406}\begin{bmatrix}139885330x_{n+1}-10y_{n+3}+69942665x_{3n+3}-5y_{3n+5}\\+13988533x_{5n+5}-y_{5n+7}\end{bmatrix}$
- $\frac{1}{1080}\begin{bmatrix}38797210x_{n+2}-29890x_{n+3}+19398605x_{3n+4}-14945x_{3n+5}\\+3879721x_{5n+6}-2989x_{5n+7}\end{bmatrix}$
- $\frac{1}{3894}\left[130x_{n+2}-29890y_{n+1}+65x_{3n+4}-14945y_{3n+3}+13x_{5n+6}-2989y_{5n+5}\right]$
- $\frac{1}{6}\begin{bmatrix}107770x_{n+2}-29890y_{n+2}+53885x_{3n+4}-14945y_{3n+4}\\+10777x_{5n+6}-2989y_{5n+6}\end{bmatrix}$
- $\frac{1}{3894}\begin{bmatrix}139885330x_{n+2}-29890y_{n+3}+69942665x_{3n+4}-14945x_{3n+5}\\13988533x_{5n+6}-2989y_{5n+7}\end{bmatrix}$
- $\frac{1}{5054406}\begin{bmatrix}130x_{n+3}-38797210y_{n+1}+65x_{3n+5}-19398605y_{3n+3}\\+13x_{5n+7}-3879721y_{5n+5}\end{bmatrix}$
- $\frac{1}{3894}\begin{bmatrix}107770x_{n+3}-38797210y_{n+2}+53885x_{3n+5}-19398605y_{3n+4}\\+10777x_{5n+7}-3879721y_{5n+6}\end{bmatrix}$
- $\frac{1}{6}\begin{bmatrix}139885330x_{n+3}-38797210y_{n+3}+69942665x_{3n+5}-19398605y_{3n+5}\\+13988533x_{5n+7}-3879721y_{5n+7}\end{bmatrix}$
- $\frac{1}{14040}\left[130y_{n+2}-107770y_{n+1}+65y_{3n+4}-53885y_{3n+3}+13y_{5n+6}-10777y_{5n+5}\right]$
- $\frac{1}{18223920}\begin{bmatrix}130y_{n+3}-139885330y_{n+1}+165y_{3n+5}-69942665y_{3n+3}\\+13y_{5n+7}-13988533y_{5n+5}\end{bmatrix}$
- $\frac{1}{14040}\begin{bmatrix}107770y_{n+3}-139885330y_{n+2}+53885y_{3n+5}-69942665y_{3n+4}\\+10777y_{5n+7}-13988533y_{5n+6}\end{bmatrix}$

REMARKABLE OBSERVATIONS:

1**.** Using the known solutions of (1), solutions in integers for different forms of hyperbolas are generated and exhibited below:

Solving (3) and (5), we get

$$f_n=\frac{1}{1080}X \quad (9)$$

$$g_n=\frac{\sqrt{13}}{1080}Y \quad (10)$$

where

$$X=2989x_{n+1}-x_{n+2}$$

$$Y = x_{n+2} - 829x_{n+1}$$

We know that

$$f_n^2 - g_n^2 = 4 \qquad (11)$$

Substituting (9) and (10) in (11), we have

$$\left[\frac{X}{1080}\right]^2 - \left[\frac{\sqrt{13}}{1080}Y\right]^2 = 4$$

$$\frac{1}{1166400}X^2 - \frac{13}{1166400}Y^2 = 4$$

$\Rightarrow X^2 - 13Y^2 = 4665600$ which represents the Hyperbola.

For elegance and clarity, the other choices of hyperbolas are presented in the Table 2 below:

Table: 2 Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$X^2 - 13Y^2 = 7860621542400$	$(3879721x_{n+1} - x_{n+3},\ x_{n+3} - 1076041x_{n+1})$
2	$X^2 - 13Y^2 = 144$	$(13x_{n+1} - y_{n+1},\ y_{n+1} - x_{n+1})$
3	$X^2 - 13Y^2 = 60652944$	$(10777x_{n+1} - y_{n+2},\ y_{n+2} - 2989x_{n+1})$
4	$X^2 - 13Y^2 = 102188080051344$	$(13988533x_{n+1} - y_{n+3,}\ y_{n+3} - 3879721x_{n+1})$
5	$X^2 - 13Y^2 = 4665600$	$(3879721x_{n+2} - 2989x_{n+3},\ 829x_{n+3} - 1076041x_{n+2})$
6	$X^2 - 13Y^2 = 60652944$	$(13x_{n+2} - 2989y_{n+1},\ 829y_{n+1} - x_{n+2})$
7	$X^2 - 13Y^2 = 144$	$(10777x_{n+2} - 2989y_{n+2},\ 829y_{n+2} - 2989x_{n+2})$
8	$X^2 - 13Y^2 = 60652944$	$(13988533x_{n+2} - 2989y_{n+3},\ 829y_{n+3} - 3879721x_{n+2})$
9	$X^2 - 13Y^2 = 102188080051344$	$(13x_{n+3} - 3879721y_{n+1},\ 1076041y_{n+1} - x_{n+3})$
10	$X^2 - 13Y^2 = 60652944$	$(10777x_{n+3} - 3879721y_{n+2},\ 1076041y_{n+2} - 2989x_{n+3})$
11	$X^2 - 13Y^2 = 144$	$(13988533x_{n+3} - 3879721y_{n+3},\ 1076041y_{n+3} - 3879721x_{n+3})$

12	$X^2-13Y^2=788486400$	$\begin{pmatrix}13y_{n+2}-10777y_{n+1}, \\ 2989y_{n+1}-y_{n+2}\end{pmatrix}$
13	$X^2-13Y^2=1328445040665600$	$\begin{pmatrix}13y_{n+3}-13988533y_{n+1}, \\ 3879721y_{n+1}-y_{n+3}\end{pmatrix}$
14	$X^2-13Y^2=788486400$	$\begin{pmatrix}10777y_{n+3}-13988533y_{n+2}, \\ 3879721y_{n+2}-2989y_{n+3}\end{pmatrix}$

2. Using the known solutions of (1), solutions in integers for different forms of parabolas are generated and exhibited below:

Replacing n by $2n+1$ in (9), we have

$$f_{2n+1}=\frac{1}{1080}\left(2989x_{2n+2}-x_{2n+3}\right)$$

Note that,

$$f_{2n+1}+2=f_n^2$$

$$\therefore\ f_n^2=\frac{1}{1080}\left(2989x_{2n+2}-x_{2n+3}\right)+2 \qquad (12)$$

Substituting (10) and (12) in (11), we have

$$\frac{1}{1080}X-\frac{13}{1166400}Y^2=4$$

$$\Rightarrow 1080X-13Y^2=4665600$$ which represents the Parabola.

For elegance and clarity, the other choice of parabolas are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(X,Y)
1	$1401840X-13Y^2=7860621542400$	$\begin{pmatrix}3879721x_{n+1}-x_{n+3}, \\ x_{n+3}-1076041x_{n+1}\end{pmatrix}$
2	$6X-13Y^2=144$	$\begin{pmatrix}13x_{2n+2}-y_{2n+2}, \\ y_{n+1}-x_{n+1}\end{pmatrix}$
3	$3894X-13Y^2=60652944$	$\begin{pmatrix}10777x_{2n+2}-y_{2n+3,} \\ y_{n+2}-2989x_{n+1}\end{pmatrix}$
4	$5054406X-13Y^2=102188080051344$	$\begin{pmatrix}13988533x_{2n+2}-y_{2n+4}, \\ y_{n+3}-3879721x_{n+1}\end{pmatrix}$
5	$1080X-13Y^2=4665600$	$\begin{pmatrix}3879721x_{2n+3}-2989x_{2n+4}, \\ 829x_{n+3}-1076041x_{n+2}\end{pmatrix}$
6	$3894X-13Y^2=60652944$	$\begin{pmatrix}13x_{2n+3}-2989y_{2n+2}, \\ 829y_{n+1}-x_{n+2}\end{pmatrix}$

7	$6X - 13Y^2 = 144$	$\begin{pmatrix} 10777x_{2n+3} - 2989y_{2n+3}, \\ 829y_{n+2} - 2989x_{n+2} \end{pmatrix}$
8	$3894X - 13Y^2 = 60652944$	$\begin{pmatrix} 13988533x_{2n+3} - 2989y_{2n+4}, \\ 829y_{n+3} - 3879721x_{n+2} \end{pmatrix}$
9	$5054406X - 13Y^2 = 144$	$\begin{pmatrix} 13x_{2n+4} - 3879721y_{2n+2}, \\ 1076041y_{n+1} - x_{n+3} \end{pmatrix}$
10	$3894X - 13Y^2 = 60652944$	$\begin{pmatrix} 10777x_{2n+4} - 3879721y_{2n+3}, \\ 1076041y_{n+2} - 2989x_{n+3} \end{pmatrix}$
11	$6X - 13Y^2 = 144$	$\begin{pmatrix} 13988533x_{2n+4} - 3879721y_{2n+4}, \\ 1076041y_{n+3} - 3879721x_{n+3} \end{pmatrix}$
12	$14040X - 13Y^2 = 788486400$	$\begin{pmatrix} 13y_{2n+3} - 10777y_{2n+2}, \\ 2989y_{n+1} - y_{n+2} \end{pmatrix}$
13	$18223920X - 13Y^2 = 1328445040665600$	$\begin{pmatrix} 13y_{2n+4} - 13988533y_{2n+2}, \\ 3879721y_{n+1} - y_{n+3} \end{pmatrix}$
14	$14040X - 13Y^2 = 788486400$	$\begin{pmatrix} 10777y_{2n+4} - 13988533y_{2n+3}, \\ 3879721y_{n+2} - 2989y_{n+3} \end{pmatrix}$

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